prepared by Professor Jenny Baglivo

Contents

4 MATH2202 Notebook 4

This notebook is concerned with topics from Sections 13.1-13.4 of the Stewart textbook.

4.1 Vector Fields in 2 Variables

Vector fields in n variables. A vector field in n variables is a vector-valued function $\boldsymbol{F}$ on a domain $\mathcal{D} \subseteq \mathbf{R}^n$ that takes values in $\mathbf{R}^n$:

$$\boldsymbol{F} : \mathcal{D} \subseteq \mathbf{R}^n \to \mathbf{R}^n$$

(the domain and range of a vector field are subsets of the <u>*same*</u> space $\mathbf{R}^n$).

> *Note* that it is common practice to use the shorthand notation for points and vectors in $\mathbf{R}^n$ when working with vector fields. In particular, we use $\boldsymbol{x}$ to represent both the point
>
> $$(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$$
>
> and the position vector from the origin to this point. And, we use the shorthand notation $\boldsymbol{F}(\boldsymbol{x})$ to represent both the point
>
> $$(F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \ldots, F_n(\boldsymbol{x})) \in \mathbf{R}^n$$
>
> and the position vector from the origin to this point, for each $\boldsymbol{x} \in \mathcal{D}$.

We will work mostly with vector fields in 2 and 3 variables.

1. <u>*Planar fields:*</u> Vector fields in 2 variables are known as planar fields.

   The usual notation for planar fields is

   $$\boldsymbol{F}(x, y) = \Big\langle P(x, y), Q(x, y) \Big\rangle = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j},$$

   where P and Q are scalar functions of 2 variables whose domains contain $\mathcal{D}$.

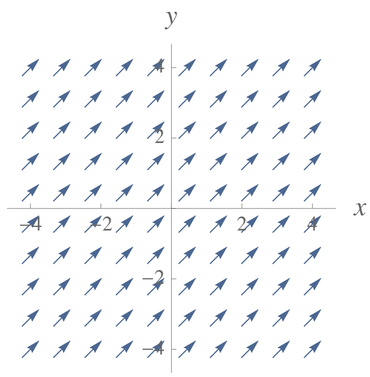
2. <u>*Spatial fields:*</u> Vector fields in 3 variables are known as spatial fields.

   The usual notation for spatial fields is

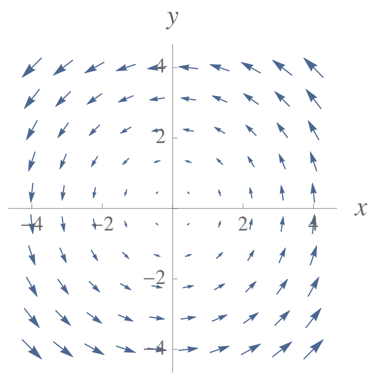
   $$\boldsymbol{F}(x, y, z) = \Big\langle P(x, y, z), Q(x, y, z), R(x, y, z) \Big\rangle = P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k},$$

   where P, Q and R are scalar functions of 3 variables whose domains contain $\mathcal{D}$.

Visualizing planar fields. Vector fields of 2 variables are visualized by drawing a vector in the direction from the point (x, y) to the point $\boldsymbol{F}(x, y)$ for various choices of $(x, y) \in \mathcal{D}$. Vectors are scaled to give the plot a "pleasing" appearance, as illustrated below:



(a) *Constant Field:* $\boldsymbol{F}(x, y) = \langle 1, 1 \rangle$



(b) *Spin Field:* $\boldsymbol{F}(x, y) = \langle -y, x \rangle$

Vectors fields are used to represent force, flow or motion. Part (a) above suggests constant movement in the northeast direction, while part (b) suggests spinning around the origin.

Gradient fields. If the vector field $\boldsymbol{F}$ has rule $\boldsymbol{F}(\boldsymbol{x}) = \nabla f(\boldsymbol{x})$ for some scalar function f, then $\boldsymbol{F}$ is said to be a gradient field. (Gradient fields are *exact derivatives* of scalar functions.)

Not all vector fields are exact derivatives. The following theorem gives a condition for determining when a planar field whose domain $\mathcal{D} = \mathbf{R}^2$ is a gradient field.

> ***Gradient Theorem for Planar Fields.*** Let $\boldsymbol{F} = \langle P, Q \rangle$ be a vector field of 2 variables with domain $\mathbf{R}^2$. If the component functions P and Q have continuous partial derivatives for all (x, y), then
>
> $$\boldsymbol{F}(x, y) \text{ is a gradient field if and only if } P_y = Q_x.$$

Note: To see why the theorem is true, note that

1. If $\boldsymbol{F}(x, y) = \nabla f(x, y)$, then $P = f_x$ and $Q = f_y$. Since P and Q have continuous partial derivatives, f has continuous second partial derivatives. Thus,

$$P_y = f_{xy} = f_{yx} = Q_x.$$

2. If $P_y = Q_x$, then we can use partial anti-differentiation to recover the family of antiderivatives. Specifically, since $P = f_x$, we know that

$$f(x, y) + C = \int P(x, y)\, dx = (\text{Expression in } x, y) + (\text{Expression in } y) + C,$$

where C is a constant. Now, use the fact that $Q = f_y$ to determine the second expression.

Exercise. In each case, determine if $\boldsymbol{F}$ is a gradient field. If $\boldsymbol{F} = \nabla f$ for some scalar function of 2 variables, then find the general form of $f(x, y)$.

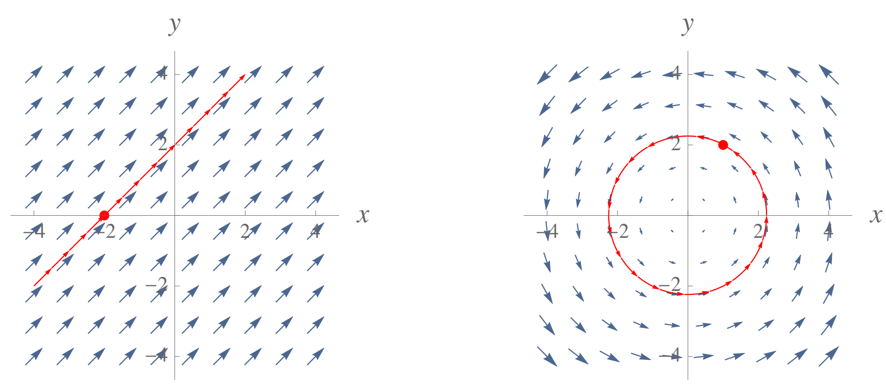
(a) $\boldsymbol{F}(x, y) = \langle 1, 1 \rangle$; (b) $\boldsymbol{F}(x, y) = \langle -y, x \rangle$.

Flow lines. A flow line (or *streamline*) for $\boldsymbol{F}(\boldsymbol{x})$ is a differentiable function $\boldsymbol{x}(t)$ satisfying

$$\boldsymbol{F}(\boldsymbol{x}(t)) = \boldsymbol{x}'(t).$$

Note: If $\boldsymbol{x}(t)$ represents position at time t, then $\boldsymbol{F}(\boldsymbol{x}(t))$ is the velocity vector at time t.

We are often interested in visualizing flow lines through particular points P in planar fields. For example, consider again the constant and spin fields from above:



(*a*) $\boldsymbol{F}(x, y) = \langle 1, 1\rangle$, $P(-2, 0)$ (*b*) $\boldsymbol{F}(x, y) = \langle -y, x\rangle$, $P(1, 2)$

Part (*a*) shows the flow of a constant field through the point $(-2, 0)$; the underlying curve is the line $y = x + 2$. Part (*b*) shows the flow of a spin field through the point $(1, 2)$; the underlying curve is the circle with equation $x^2 + y^2 = 5$.

Finding underlying curves. The underlying curve of a flow line in planar fields can often be found by using separation of variables. To see this, let $\boldsymbol{x}(t) = (x(t), y(t))$. Then

$$\boldsymbol{F}(\boldsymbol{x}(t)) = \boldsymbol{x}'(t) \quad \Rightarrow \quad P(x, y) = \frac{dx}{dt} \quad \text{and} \quad Q(x, y) = \frac{dy}{dt}.$$

Now,

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q(x, y)}{P(x, y)} \quad \text{when } P(x, y) \neq 0.$$

If the variables in the ratio Q/P can be separated, that is, if

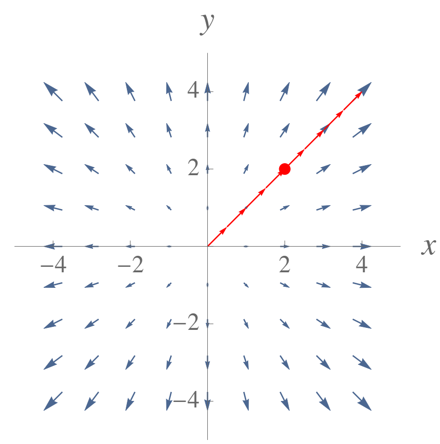
$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} \quad \Rightarrow \quad (\text{Expression in } y)\ dy = (\text{Expression in } x)\ dx,$$

then the left and right sides can be integrated separately to find a general description of the underlying curve in x and y only.

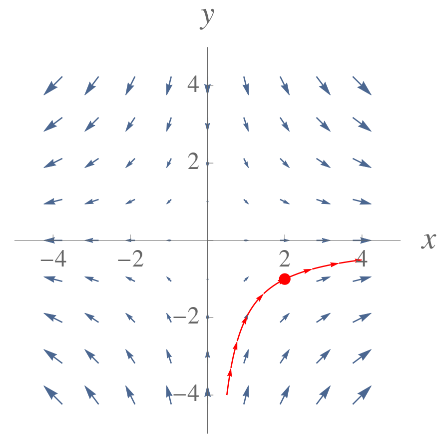
Exercise. In each case,

1. Determine if $\boldsymbol{F}$ is a gradient field;
2. If $\boldsymbol{F}$ is a gradient field, find its family of antiderivatives;
3. Find a formula for the underlying curve of the flow line through the given point P.

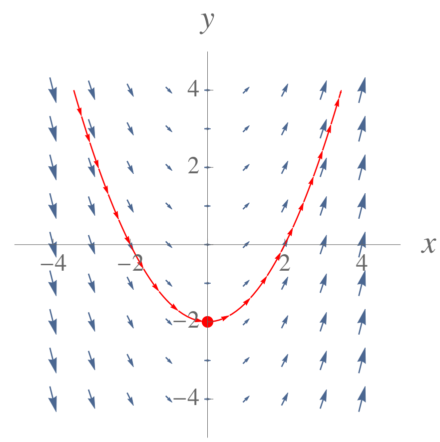
(a) $\boldsymbol{F}(x, y) = x\boldsymbol{i} + y\boldsymbol{j}$, $P(2, 2)$



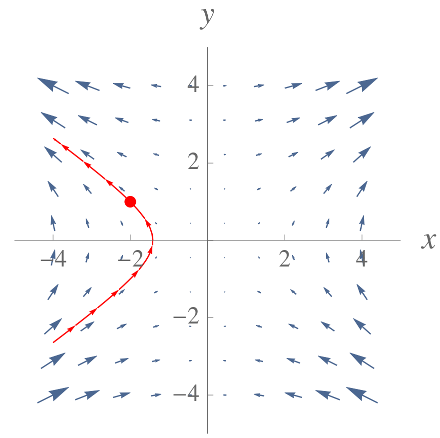
(b) $\boldsymbol{F}(x, y) = x\boldsymbol{i} - y\boldsymbol{j},\ P(2, -1)$



(c) $\boldsymbol{F}(x, y) = \boldsymbol{i} + x\boldsymbol{j},\ P(0, -2)$



(d) $\boldsymbol{F}(x,y) = 2xy\boldsymbol{i} + x^2\boldsymbol{j},\ P(-2,1)$



Gradient fields, flow lines and level curves. If $\boldsymbol{F}(x,y) = \nabla f(x,y)$ is a gradient field, then the level curve containing the point (a,b) is orthogonal to the flow line through (a,b).

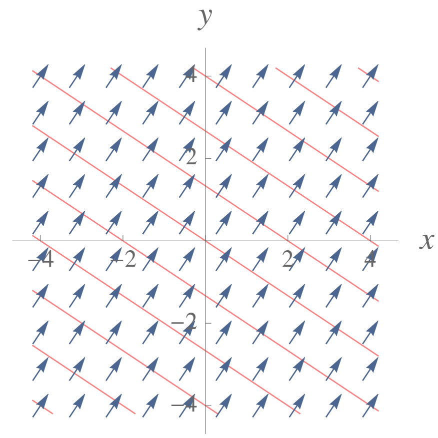
Exercise. Let f be the scalar function of 2 variables with rule

$$f(x,y) = 2x + 3y - 5.$$

The plot to the right is a contour map of f with heights

$$k = -25, -21, \ldots, 15,$$

and with the gradient field $\boldsymbol{F} = \nabla f$ superimposed.



(a) Find a general form for the level curves of f.

(b) Find a general form for the flow lines of $\boldsymbol{F} = \nabla f$.

(c) Explain why the level curve containing (a,b) and the flow line through (a,b) are orthogonal, for all (a,b).

4.2 Vector Fields in 3 Variables

Visualizing spatial fields. Vector fields in 3 variables,

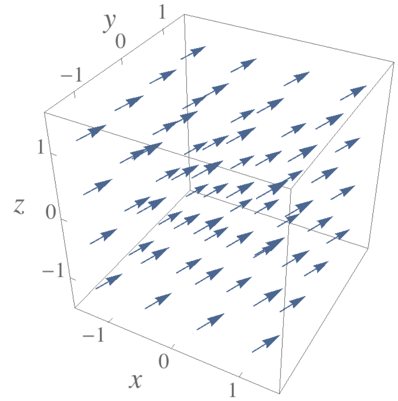
$$\boldsymbol{F}(x,y,z) = \Big\langle P(x,y,z), Q(x,y,z), R(x,y,z) \Big\rangle = P(x,y,z)\boldsymbol{i} + Q(x,y,z)\boldsymbol{j} + R(x,y,z)\boldsymbol{k},$$

are visualized by drawing a vector in the direction from the point (x,y,z) to the point $\boldsymbol{F}(x,y,z)$ for various choices of (x,y,z) in the domain, where a scaling factor is applied to give the plot a "pleasing" appearance.

For example, the plot to the right illustrates the constant field with rule

$$\boldsymbol{F}(x,y,z) = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}.$$

[Although the field is constant, the vectors in the foreground appear to be larger than the vectors in the background.]



Gradient fields. As before, not all spatial fields can be viewed as the exact derivative of a scalar function. The following theorem gives conditions for determining when a spatial field whose domain $\mathcal{D} = \mathbf{R}^3$ is a gradient field.

> ***Gradient Theorem for Spatial Fields.*** Let $\boldsymbol{F} = \langle P, Q, R\rangle$ be a vector field of 3 variables with domain $\mathbf{R}^2$. If the component functions P, Q and R have continuous second partial derivatives for all (x,y,z), then
>
> $\boldsymbol{F}(x,y,z)$ is a gradient field if and only if $P_y = Q_x$, $P_z = R_x$ and $Q_z = R_y$.

Note: To see why the theorem is true, note that

1. If $\boldsymbol{F}(x,y,z) = \nabla f(x,y,z)$, then $P = f_x$, $Q = f_y$ and $R = f_z$. Since P, Q and R have continuous partial derivatives, f has continuous second partial derivatives. Thus,

   (a) $P_y = f_{xy} = f_{yx} = Q_x$,

   (b) $P_z = f_{xz} = f_{zx} = R_x$,

   (c) $Q_z = f_{yz} = f_{zy} = R_y$.

2. If $P_y = Q_x$, $P_z = R_x$ and $Q_z = R_y$, then partial anti-differentiation can be used to recover the family of antiderivatives, as illustrated below.

3. To illustrate the result of the gradient theorem, consider again the constant field

$$\boldsymbol{F}(x, y, z) = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}.$$

Since all partial derivatives are zero, the conditions of the theorem are satisfied.

Now,

(1) Since $f_x(x, y, z) = P(x, y, z) = 2$, we know that

$$f(x, y, z) = 2x + g(y, z) \text{ for some scalar function } g(y, z),$$

and $\frac{\partial}{\partial y}(f(x, y, z)) = \frac{\partial}{\partial y}(2x + g(y, z)) = g_y(y, z)$.

(2) Since $f_y(x, y, z) = Q(x, y, z) = 3$, the last result tells us that

$$3 = g_y(y, z) \;\Rightarrow\; g(y, z) = 3y + h(z) \text{ for some scalar function } h(z),$$

and $\frac{\partial}{\partial z}(f(x, y, z)) = \frac{\partial}{\partial z}(2x + 3y + h(z)) = h'(z)$.

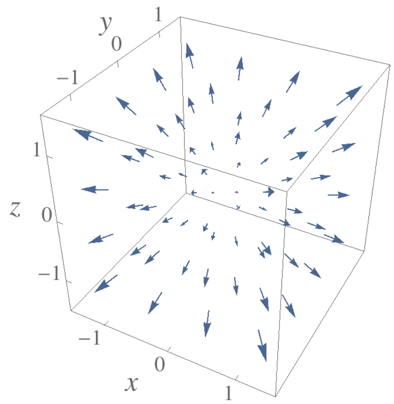
(3) Since $f_z(x, y, z) = R(x, y, z) = 1$, the last result tells us that

$$1 = h'(z) \;\Rightarrow\; h(z) = z + C \text{ for some constant } C.$$

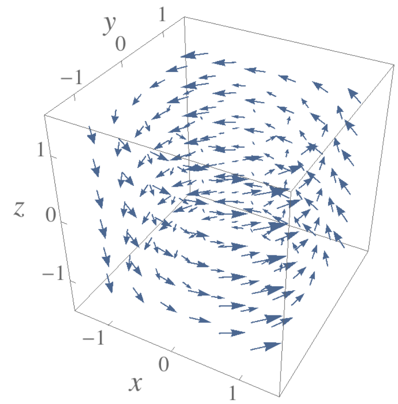
Thus, the family of antiderivatives is $f(x, y, z) = 3x + 2y + z + C$.

Exercise. In each case shown on the next pages,

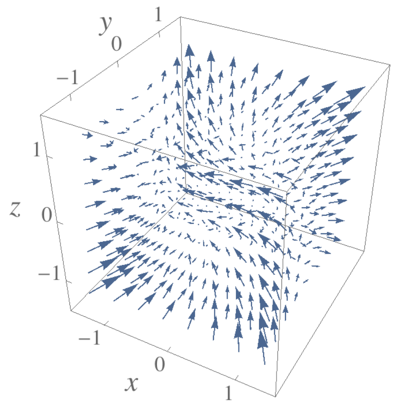
1. Determine if $\boldsymbol{F}$ is a gradient field.
2. If $\boldsymbol{F}$ is a gradient field, find its family of antiderivatives.



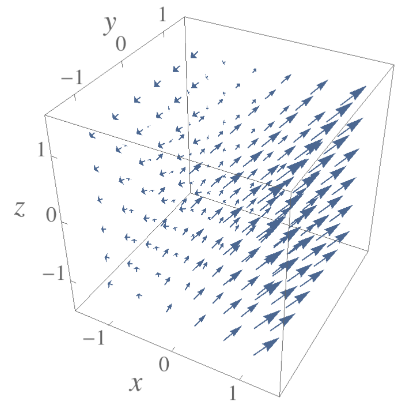
(a) $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$



(b) $\boldsymbol{F}(x, y, z) = -y\boldsymbol{i} + x\boldsymbol{j}$



(c) $\boldsymbol{F}(x, y, z) = 2xy\boldsymbol{i} + (x^2 + 2yz)\boldsymbol{j} + y^2\boldsymbol{k}$



(d) $\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + \boldsymbol{j} + x\boldsymbol{k}$

4.3 Smooth Curves, Line Integrals

Integration over intervals can be generalized to integration over smooth curves.

1. *Smooth Curve:* The curve $\mathcal{C}$ in n-space is said to be smooth if it can be parametrized using a differentiable vector function

$$\boldsymbol{r}(t), \text{ for } t \in [a,b], \text{ for which } \boldsymbol{r'}(t) \neq \boldsymbol{O} \text{ throughout the interval.}$$

2. *Line Integral:* Let f be a continuous function of n variables whose domain contains the smooth curve $\mathcal{C}$, and let $\boldsymbol{r}(t)$, $t \in [a,b]$, be a smooth parametrization of $\mathcal{C}$ that traverses the curve exactly once as t increases from a to b, except possibly at the endpoints.

   Then, the line integral of f along $\mathcal{C}$ is defined as follows:

$$\int_{\mathcal{C}} f \; ds = \int_a^b f(\boldsymbol{r}(t))|\boldsymbol{r'}(t)| \; dt.$$

   In this formula, s represents arc length and $ds = |\boldsymbol{r'}(t)| \; dt$ is a length element.

Notes:

1. Recall from earlier this term that if

   (*i*) $\boldsymbol{r}(t)$, for $t \in [a,b]$, is a differentiable vector function,

   (*ii*) $\boldsymbol{r'}(t) \neq \boldsymbol{O}$ throughout the interval, and

   (*iii*) the curve underlying $\boldsymbol{r}(t)$ is traversed exactly once as t increases from a to b,

   then the length of the underlying curve can be computed using the formula

$$L = \int_a^b |\boldsymbol{r'}(t)| \; dt.$$

2. The length formula can be viewed as a special case of the formula for line integrals:

$$L = \int_{\mathcal{C}} 1 \; ds = \int_a^b |\boldsymbol{r'}(t)| \; dt.$$

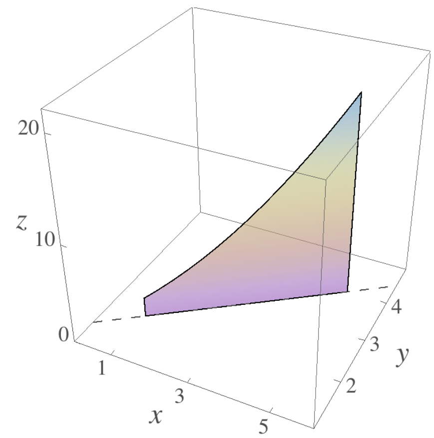
   When t is a time parameter, arc length can be interpreted as total distance travelled; the total distance travelled is obtained by integrating the speed over time.

3. Smooth curves can be parametrized in many different ways. But, the value of the line integral of f along $\mathcal{C}$ remains the same as long as a smooth parametrization is used that traverses $\mathcal{C}$ exactly once, except possibly at its endpoints.

Exercise. In each case, evaluate $\int_{\mathcal{C}} f\ ds$ using the given parametrization.

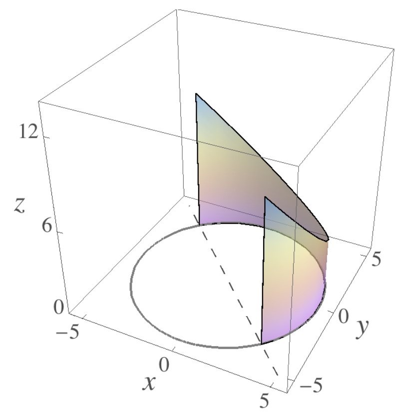
(*a*) $f(x, y) = xy$ and

$$\mathcal{C}: \ \boldsymbol{r}(t) = \langle 1 + 4t, 2 + 2t \rangle, \ t \in [0, 1]$$



(*b*) $f(x, y) = 10 - x - y$ and

$$\mathcal{C}: \langle 5\cos(t), 5\sin(t)\rangle, \ t \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$$



Exercise. The half-circle $\mathcal{C} = \{(x, y) \ : \ x^2 + y^2 = 1, \ y \geq 0\}$ can be oriented in a

1. *Counterclockwise direction* using the vector function $\boldsymbol{r}(t) = \langle \cos(t), \sin(t) \rangle$, $t = [0, \pi]$.
2. *Clockwise direction* using the vector function $\boldsymbol{r}(t) = \langle \cos(\pi - t), \sin(\pi - t) \rangle$, $t = [0, \pi]$.

Demonstrate that $\int_{\mathcal{C}} 2y \ ds$ remains the same no matter which way $\mathcal{C}$ is oriented.

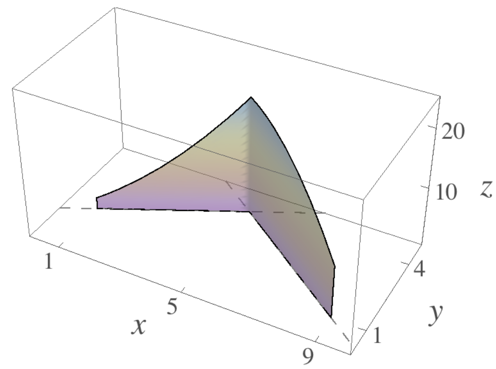
Exercise. The formula for line integrals extends to piecewise smooth curves by taking the sum of the integrals over each piece. Use this fact to evaluate

$$\int_{\mathcal{C}} xy\ ds,$$

where $\mathcal{C}$ is the piecewise linear curve obtained by connecting the points

$$(1,2),\ (5,4),\ (9,1)$$

successively.



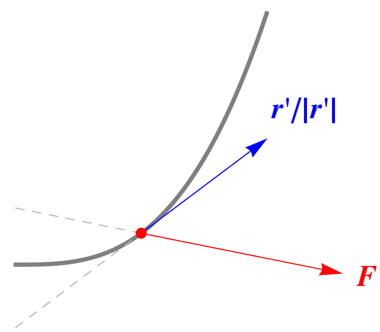
Line integrals of vector fields. Let $\boldsymbol{F}$ be a continuous vector field whose domain contains the smooth curve $\mathcal{C}$, and let $\boldsymbol{r}(t)$, for $t \in [a, b]$, be a smooth parametrization of $\mathcal{C}$ that traverses the curve exactly once as t increases from a to b, except possibly at the endpoints.

Then the line integral of $\boldsymbol{F}$ along $\mathcal{C}$ is defined as follows:

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\ dt,$$

where the integrand on the right is the dot product of $\boldsymbol{F}(\boldsymbol{r}(t))$ and $\boldsymbol{r}'(t)$.

Notes: Line integrals of vector fields are a special case of line integrals of scalar functions.



To understand the relationship better, let $\boldsymbol{x} = \boldsymbol{r}(t)$ be a point on the curve, and

$$g(\boldsymbol{x}) = g(\boldsymbol{r}(t))$$

be the dot product of $\boldsymbol{F}(\boldsymbol{x})$ with the unit tangent vector pointing in the direction of increasing t at this point,

$$g(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \left(\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} \right) = |\boldsymbol{F}(\boldsymbol{r}(t))| \cos(\theta),$$

where θ is the angle between $\boldsymbol{F}(\boldsymbol{r}(t))$ and $\boldsymbol{r}'(t)$.

Now,

$$\int_{\mathcal{C}} g\ ds = \int_a^b g(\boldsymbol{r}(t))|\boldsymbol{r}'(t)|\ dt = \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\ dt = \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

The importance of this special case can be seen by observing that $g(\boldsymbol{x})$ assumes its

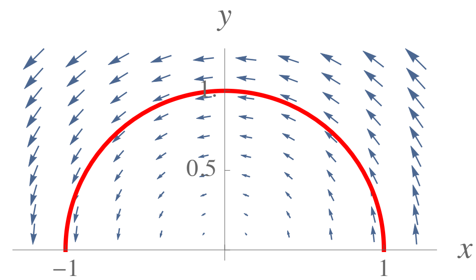
1. *Minimum* value when the direction of flow of the vector field is *opposite* the direction of increasing t along $\mathcal{C}$, and
2. *Maximum* value when the direction of flow of the vector field is *same* as the direction of increasing t along $\mathcal{C}$.

Thus, the line integral of $\boldsymbol{F}$ over $\mathcal{C}$ measures the extent to which you are "going with the flow".

Exercise. Evaluate $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F}$ has rule

$$\boldsymbol{F}(x, y) = -y\boldsymbol{i} + x\boldsymbol{j},$$

and $\mathcal{C}$ is the half-circle shown to the right.

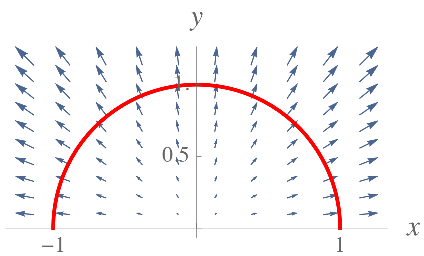


(a) Use $\boldsymbol{r}(t) = \langle \cos(t), \sin(t) \rangle$, for $t \in [0, \pi]$, to parametrize $\mathcal{C}$.

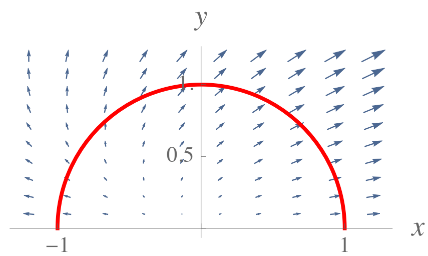
(b) Use $\boldsymbol{r}(t) = \langle \cos(\pi - t), \sin(\pi - t) \rangle$, for $t \in [0, \pi]$, to parametrize $\mathcal{C}$.

Exercise. In each case, evaluate $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}$ over the half-circle $\mathcal{C}$ parametrized as follows:

$$\mathcal{C} \;:\; \boldsymbol{r}(t) = \langle \cos(t), \sin(t) \rangle, \;\; t \in [0, \pi].$$



(a) $\boldsymbol{F}(x, y) = x\boldsymbol{i} + y\boldsymbol{j}$,



(b) $\boldsymbol{F}(x, y) = (x + y)\boldsymbol{i} + y\boldsymbol{j}$,

4.4 Fundamental Theorem of Calculus (FTC) for Line Integrals

The Fundamental Theorem of Calculus for integrals over intervals can be generalized to integrals over smooth curves. Specifically,

> ***Fundamental Theorem of Calculus for Line Integrals.*** Let f be a continuous function of n variables with continuous first partial derivatives on a domain containing the smooth curve $\mathcal{C}$, and let $\boldsymbol{r}(t)$, $t \in [a, b]$, be a smooth parametrization of $\mathcal{C}$ that traverses the curve exactly once as t increases from a to b, except possibly at the endpoints. Then
>
> $$\int_{\mathcal{C}} (\nabla f) \cdot d\boldsymbol{r} = f(\boldsymbol{r}(b)) - f(\boldsymbol{r}(a)).$$

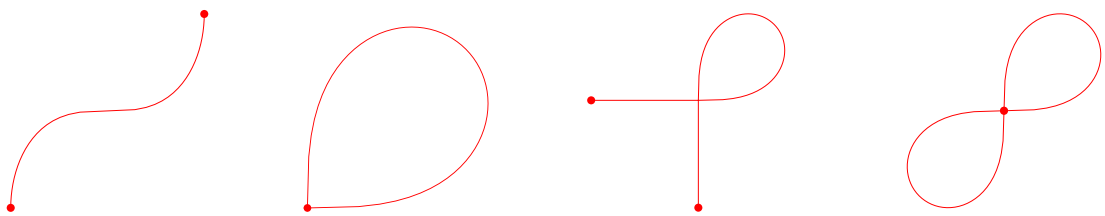
Exercise. Let f be the function of 3 variables with rule $f(x, y, z) = x^2 - y^2 + z^2$ and let $\mathcal{C}$ be the directed line segment from $(0, 5, 0)$ to $(1, 3, 4)$.

(a) Use the definition of line integral to evaluate $\int_{\mathcal{C}} (\nabla f) \cdot d\boldsymbol{r}$.

(b) Re-evaluate the result using the fundamental theorem of calculus for line integrals.

Focus on curves. If the curve $\mathcal{C}$ begins and ends at the same point, then $\mathcal{C}$ is said to be a *closed curve*. If the curve $\mathcal{C}$ has no self-intersections, except possibly at its endpoints, then $\mathcal{C}$ is said to be a *simple curve*.

The following examples illustrate the four possibilities:



(1) *Simple* (2) *Closed, Simple* (3) *Not Simple* (4) *Closed, Not Simple*

The fundamental theorem of calculus for line integrals applies to simple smooth curves, and can be generalized to simple piecewise-smooth curves. In addition,

1. Closed Curve: If $\mathcal{C}$ is a closed, simple, piecewise-smooth curve, then

$$\int_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r} = 0,$$

when $\boldsymbol{F} = \nabla f$ is a continuous gradient field whose domain contains $\mathcal{C}$.

2. Path Independence: If $\mathcal{C}_1$ and $\mathcal{C}_2$ are simple piecewise-smooth curves that begin and end at the same points, then

$$\int_{\mathcal{C}_1} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{\mathcal{C}_2} \boldsymbol{F}\cdot d\boldsymbol{r},$$

when $\boldsymbol{F} = \nabla f$ is a continuous gradient field whose domain contains both curves.

Note: A natural question to ask is the following:

> "Let $\boldsymbol{F}$ be a continuous vector field on the domain $\mathcal{D}$, and suppose that the value of the line integral $\int_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r}$ is the same for all simple piecewise-smooth curves beginning and ending at the same points A and B, for arbitrarily chosen $A, B \in \mathcal{D}$. Is $\boldsymbol{F}$ actually a gradient field?"

The theorems at the beginning of this notebook tell us that the answer is "yes" for planar fields where $\mathcal{D} = \mathbf{R}^2$ (page 4) and for spatial fields where $\mathcal{D} = \mathbf{R}^3$ (page 12).

These theorems can be generalized to domains that are *simply-connected*, that is, to regions D that are connected and satisfy the property that every simple closed curve in D encloses only points in D (see Section 13.3 of the textbook).

Alternative notations. We sometimes use alternative notations for $\boldsymbol{F} \cdot d\boldsymbol{r}$. For example,

1. <u>*For planar fields,*</u>

$$\boldsymbol{F} \cdot d\boldsymbol{r} = \langle P(x,y), Q(x,y) \rangle \cdot \langle dx, dy \rangle = P(x,y)\ dx + Q(x,y)\ dy.$$

2. <u>*For spatial fields,*</u>

$$\boldsymbol{F} \cdot d\boldsymbol{r} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle \cdot \langle dx, dy, dz \rangle = P(x,y,z)\ dx + Q(x,y,z)\ dy + R(x,y,z)\ dz.$$

The expressions on the right are examples of *differential forms*. When $\boldsymbol{F} = \nabla f$ is a gradient field, then the differential form reduces to the *exact differential*: $\boldsymbol{F} \cdot d\boldsymbol{r} = df$.

Note: A shorthand for the FTC for line integrals is

$$\int_{\mathcal{C}} (\nabla f) \cdot d\boldsymbol{r} = \int_{\mathcal{C}} df = f(B) - f(A),$$

where A and B are the initial points and terminal of $\mathcal{C}$ in the given parametrization.

4.5 Green's Theorem

The double integral over a planar region can sometimes be found by computing a line integral over the boundary of the region.

Boundary orientations for simple closed curves. Let $D \subset \mathbf{R}^2$ be a closed and bounded set whose boundary is the simple closed curve $\mathcal{C} = \partial D$. $\mathcal{C}$ is said to have a *positive orientation* if the domain D is on the left, and to have a *negative orientation* if the domain on the right, as you traverse $\mathcal{C}$.

For example, if D is the triangular region with corners $(0,0)$, $(0,1)$, $(1,0)$, then

(1) The piecewise linear curve traversing the corners in the following order

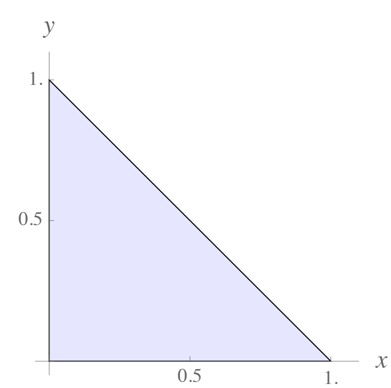
$$(0,0) \longrightarrow (1,0) \longrightarrow (0,1) \longrightarrow (0,0)$$

gives a positive orientation to the boundary.

(2) The piecewise linear curve traversing the corners in the following order

$$(0,0) \longrightarrow (0,1) \longrightarrow (1,0) \longrightarrow (0,0)$$

gives a negative orientation to the boundary.



Green's theorem. The following theorem, discovered by George Green in the 1820's, gives a relationship between a line integral around a simple closed curve in the plane and a double integral over the planar region bounded by the curve.

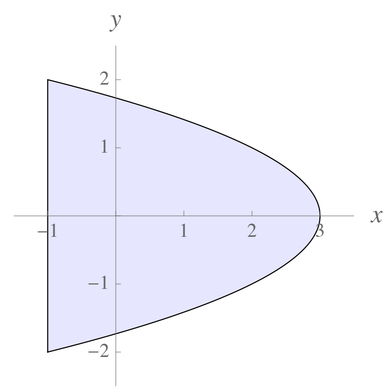
> ***Green's Theorem.*** Let $D \subset \mathbf{R}^2$ be a closed and bounded set with no holes and let $\mathcal{C}$ be the boundary of D. Assume that $\mathcal{C}$ is a simple closed curve with positive orientation. Let $\boldsymbol{F} = \langle P, Q \rangle$ be a continuous planar field whose domain includes a neighborhood around each point of D. Then
>
> $$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = \oint_{\mathcal{C}} (P\ dx + Q\ dy) = \iint_D (Q_x - P_y)\ dA,$$
>
> where the symbol $\oint$ is used to indicate that you are integrating over a closed curve.

Exercise. Let $\mathcal{C}$ be the boundary of the region where

$$x \leq 3 - y^2 \text{ and } -2 \leq y \leq 2,$$

oriented positively. $\mathcal{C}$ is the union of two simple curves:

1. $\mathcal{C}_1$: The parabola $x = 3 - y^2$ from $(-1, -2)$ to $(-1, 2)$.
2. $\mathcal{C}_2$: The directed line segment from $(-1, 2)$ to $(-1, -2)$.



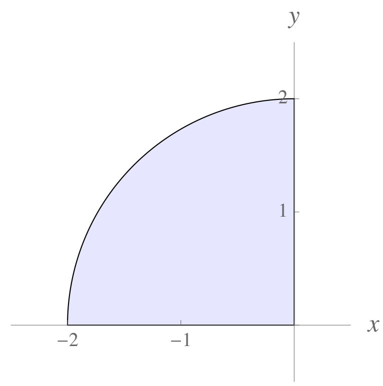
(a) Use Green's Theorem to evaluate $\oint_{\mathcal{C}}(1\ dx + 2x\ dy)$.

(b) Check your answer by evaluating $\int_{\mathcal{C}_1}(1\ dx + 2x\ dy) + \int_{\mathcal{C}_2}(1\ dx + 2x\ dy)$ directly.

Exercise. Let $\mathcal{C}$ be the boundary of the region where

$$x^2 + y^2 \leq 4,\ x \leq 0,\ y \geq 0,$$

oriented positively. $\mathcal{C}$ is the union of three simple curves:



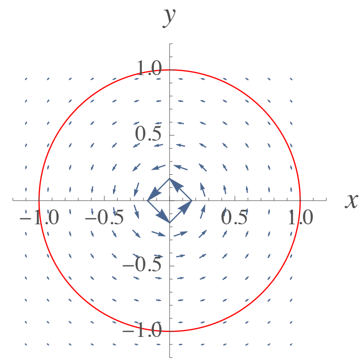
1. $\mathcal{C}_1$: The directed line segment from $(-2, 0)$ to $(0, 0)$.
2. $\mathcal{C}_2$: The directed line segment from $(0, 0)$ to $(0, 2)$.
3. $\mathcal{C}_3$: The part of the circle $x^2 + y^2 = 4$ in the second quadrant from $(0, 2)$ to $(-2, 0)$.

(a) Use Green's Theorem to evaluate $\oint_{\mathcal{C}} ((x^3 - y^3)\ dx + (x^3 + y^3)\ dy)$.

(b) Evaluate $\int_{\mathcal{C}_3} ((x^3 - y^3)\ dx + (x^3 + y^3)\ dy)$ by using your answer to part (a) and line integrals over the two directed line segments.

Exercise. Let D be the unit disk $x^2 + y^2 \leq 1$, let $\mathcal{C}$ be the boundary of the disk oriented positively, and let

$$\boldsymbol{F}(x, y) = \left(\frac{-y}{x^2 + y^2}\right) \boldsymbol{i} + \left(\frac{x}{x^2 + y^2}\right) \boldsymbol{j}.$$



Note that, since $\boldsymbol{F}$ is not defined at the origin, Green's theorem cannot be used to find the line integral along $\mathcal{C}$.

Demonstrate that

$$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} \neq \iint_D \left(Q_x - P_y\right) dA$$

by showing that the values of the two integrals are not equal.

Computing areas. If $Q_x - P_y = 1$, then the area of the domain D can be computed using a line integral around the boundary.

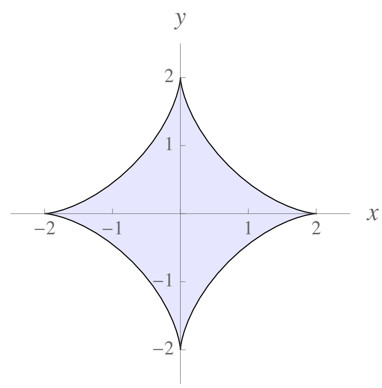
Three commonly used choices are

$$\text{Area} = \oint_{\mathcal{C}} (x\ dy) = \oint_{\mathcal{C}} (-y\ dx) = \frac{1}{2} \oint_{\mathcal{C}} (-y\ dx + x\ dy).$$

Exercise. Use Green's theorem to find the area enclosed by the parametrized curve

$$\boldsymbol{r}(t) = \langle 2\cos^3(t), 2\sin^3(t) \rangle, \text{ for } t \in [0, 2\pi].$$

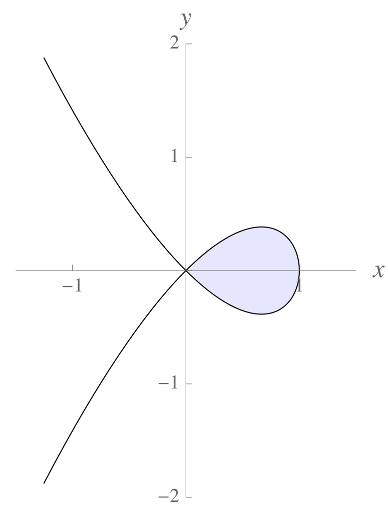
This curve is known as a *hypocycloid*.



Exercise. The underlying curve of

$$\boldsymbol{r}(t) = \langle 1 - t^2, t^3 - t \rangle, \text{ for } t \in [-3/2, 3/2],$$

is shown to the right. Use Green's theorem to find the area inside the closed loop shown in the plot.



4.6 Beyond Green's Theorem

***Tangential and outward normal components of* F.** As in Green's theorem, let $D \subset \mathbf{R}^2$ be a closed and bounded set with no holes and let $\mathcal{C}$ be the boundary of D. Assume that $\mathcal{C}$ is a simple closed curve with positive orientation. Let $\boldsymbol{F} = \langle P, Q \rangle$ be a continuous planar field whose domain includes a neighborhood around each point of D.

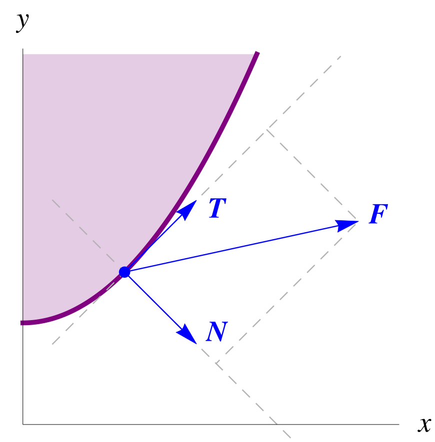
Let $\boldsymbol{r}(t) = \langle x(t), y(t) \rangle$ be a differentiable path whose image is the curve $\mathcal{C}$ with positive orientation, and with nonzero derivative. Then

1. The unit tangent pointing along the curve is

$$\boldsymbol{T}(t) = \frac{1}{|\boldsymbol{r'}(t)|} \langle x'(t), y'(t) \rangle.$$

2. The *outward* pointing unit normal is

$$\boldsymbol{N}(t) = \frac{1}{|\boldsymbol{r'}(t)|} \langle -y'(t), x'(t) \rangle.$$



For each t, $(\boldsymbol{F} \cdot \boldsymbol{T})$ is the scalar multiple of the projection of $\boldsymbol{F}$ in the direction of the curve. Similarly, for each t, $(\boldsymbol{F} \cdot \boldsymbol{N})$ is the scalar multiple of the projection of $\boldsymbol{F}$ in the direction of the outward pointing normal.

Green's theorem implies that

$$\int_{\mathcal{C}} (\boldsymbol{F} \cdot \boldsymbol{T}) \; ds = \int_{\mathcal{C}} (P \; dx + Q \; dy) = \iint_D (Q_x - P_y) \; dA$$

and

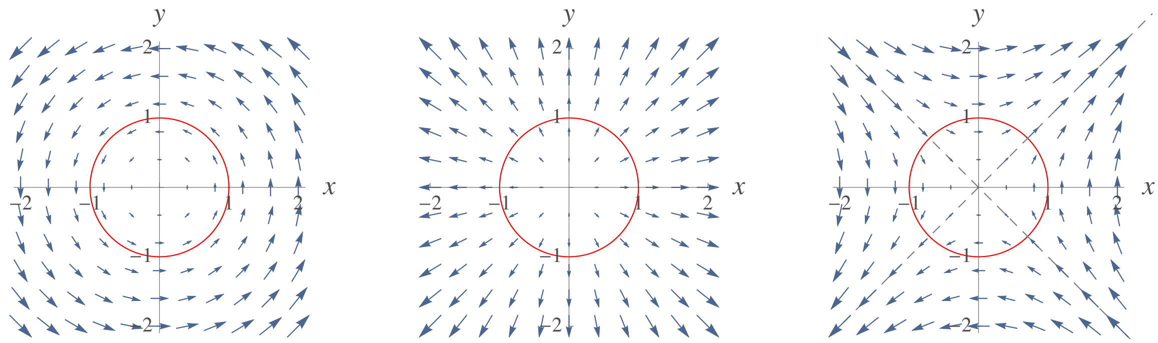
$$\int_{\mathcal{C}} (\boldsymbol{F} \cdot \boldsymbol{N}) \; ds = \int_{\mathcal{C}} (-Q \; dx + P \; dy) = \iint_D (P_x + Q_y) \; dA.$$

Curl, divergence. There are two new types of derivatives associated with planar fields.

1. Curl: The curl of the planar field $\boldsymbol{F}$ is the quantity $(Q_x - P_y)$ used in the evaluation of the line integral of the tangential component of $\boldsymbol{F}$.

2. Divergence: The divergence of the planar field $\boldsymbol{F}$ is the quantity $(P_x + Q_y)$ used in the evaluation of the line integral of the outward normal component of $\boldsymbol{F}$.

Illustrations. The curl is related to the rate of rotation (or spin) of a field around a point. The divergence is related to the rate of expansion of objects in the field (or outward flow from a source). The three planar fields below can be used to illustrate these two derivative methods.



	$\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j}$	**Curl:** $Q_x - P_y$	**Divergence:** $P_x + Q_y$
Left:	$-y\boldsymbol{i} + x\boldsymbol{j}$		
Middle:	$x\boldsymbol{i} + y\boldsymbol{j}$		
Right:	$y\boldsymbol{i} + x\boldsymbol{j}$		

Green's theorem in 3-space. A 3-dimensional version of Green's theorem can be developed to relate the double integral of a differential form over a parametrized surface $\mathcal{S}$ to the triple integral of a scalar function of 3 variables over the region bounded by $\mathcal{S}$:

$$\iint_{\mathcal{S}} (P\ dy\ dz + Q\ dz\ dx + R\ dx\ dy) = \iiint_E (P_x + Q_y + R_z)\ dV.$$

The boundary surface $\mathcal{S} = \partial E$ needs to be oriented "positively" (that is, the unit normals need to point away from the interior of the domain E), and the field $F = \langle P, Q, R \rangle$ needs to be a continuous spatial field on an open set containing the domain E.