MATH2210 Notebook 1 — Spring 2018

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1 MATH2210 Notebook 1

This notebook is concerned with introductory linear algebra concepts. The notes correspond to material in Chapter 1 of the Lay textbook.

1.1 Solving Systems of Linear Equations

In linear algebra we study linear equations and systems of linear equations.

Linear algebra methods are used to solve problems in areas as diverse as ecology (e.g. population projections), economics (e.g. input-output analysis), engineering (e.g. analysis of air flow), and computer graphics (e.g. perspective drawing). Further, linear methods are fundamental in statistical analysis of multivariate data.

1.1.1 Linear Equations and Linear Systems

A *linear equation* in the variables x_1, x_2, ..., x_n is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n \; = \; b$$

where $a_1, a_2, \ldots, a_n$ and b are constants. A *system of linear equations* in the variables x_1, x_2, ..., x_n is a collection of one or more linear equations in these variables. For example,

$$\begin{array}{rrrcr} x_1 & -2x_2 & +\,x_3 & = & 0 \\ & 2x_2 & -8x_3 & = & 8 \\ -4x_1 & +5x_2 & +9x_3 & = & -9 \end{array}$$

is a system of 3 linear equations in the 3 unknowns x_1, x_2, x_3 (a "3-by-3 system").

1.1.2 Solutions and Solution Sets

A *solution* of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when s_i is substituted for x_i for $i = 1, 2, \ldots, n$. The list (29, 16, 3) is a solution to the system above. To check this, note that

$$\begin{array}{rrrcr} (29) & -2(16) & +\,(3) & = & 0 \\ & 2(16) & -8(3) & = & 8 \\ -4(29) & +5(16) & +9(3) & = & -9 \end{array}$$

The *solution set* of a linear system is the collection of all possible solutions of the system. For the example above, (29, 16, 3) is the unique solution.

Two linear systems are *equivalent* if each has the same solution set.

1.1.3 Elementary Row Operations, Forward and Backward Pass

The strategy for solving a linear system is to replace the system with an equivalent one that is easier to solve. The equivalent system is obtained using *elementary row operations*:

1. (*Replacement*) Add to one row a multiple of another,
2. (*Interchange*) Interchange two rows,
3. (*Scaling*) Multiply all entries in a row by a nonzero constant,

where each "row" corresponds to an equation in the system.

Example 1. We work out the strategy using the 3-by-3 system given above, where a 3-by-4 *augmented matrix* of the coefficients and right hand side values follows our progress toward a solution.

$$(1)\qquad \begin{array}{rrrrcr} R_1: & x_1 & -2x_2 & +x_3 & = & 0 \\ R_2: & & 2x_2 & -8x_3 & = & 8 \\ R_3: & -4x_1 & +5x_2 & +9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$(2)\qquad \begin{array}{rrrrcr} R_1: & x_1 & -2x_2 & +x_3 & = & 0 \\ R_2: & & 2x_2 & -8x_3 & = & 8 \\ R_3+4R_1: & & -3x_2 & +13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$(3)\qquad \begin{array}{rrrrcr} R_1: & x_1 & -2x_2 & +x_3 & = & 0 \\ \frac{1}{2}R_2: & & x_2 & -4x_3 & = & 4 \\ R_3: & & -3x_2 & +13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$(4)\qquad \begin{array}{rrrrcr} R_1: & x_1 & -2x_2 & +x_3 & = & 0 \\ R_2: & & x_2 & -4x_3 & = & 4 \\ R_3+3R_2: & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$(5)\qquad \begin{array}{rrrrcr} R_1-R_3: & x_1 & -2x_2 & & = & -3 \\ R_2+4R_3: & & x_2 & & = & 16 \\ R_3: & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$(6)\qquad \begin{array}{rrrrcr} R_1+2R_2: & x_1 & & & = & 29 \\ R_2: & & x_2 & & = & 16 \\ R_3: & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Thus, the solution is $x_1 = 29$, $x_2 = 16$, and $x_3 = 3$.

Forward and backward phases. In the *forward phase* of the row reduction process (systems (1) through (4)), R_1 is used to eliminate x_1 from R_2 and R_3; then R_2 is used to eliminate x_2 from R_3. In the *backward phase* of the row reduction process (systems (5) and (6)), R_3 is used to eliminate x_3 from R_1 and R_2; then R_2 is used to eliminate x_2 from R_1.

Footnote: Gaussian and Gauss-Jordan Elimination

The forward phase is often referred to as *Gaussian elimination.* If both forward and backward phases are done, the process is often called *Gauss-Jordan elimination.*

1.1.4 Existence and Uniqueness

Two fundamental questions about linear systems are:

1. Does a solution exist?
2. If a solution exists, is it unique?

If a linear system has one or more solutions, then it is said to be *consistent*; otherwise, it is said to be *inconsistent.* A linear system can be inconsistent, or have a unique solution, or have infinitely many solutions.

Note that in the example above, we knew that the system was consistent once we reached equivalent system (4); the remaining steps allowed us to find the unique solution.

Example 2. The following 3-by-3 linear system has an infinite number of solutions:

$$\begin{array}{rrrcr} 2x_1 & +4x_2 & +2x_3 & = & 16 \\ 3x_1 & +7x_2 & +2x_3 & = & 29 \\ -4x_1 & -12x_2 & & = & -52 \end{array}$$

To demonstrate this, consider the sequence of equivalent augmented matrices:

$$\left[\begin{array}{rrr|r} 2 & 4 & 2 & 16 \\ 3 & 7 & 2 & 29 \\ -4 & -12 & 0 & -52 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 1 & 8 \\ 0 & 1 & -1 & 5 \\ 0 & -4 & 4 & -20 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 1 & 8 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix corresponds to $x_1 + 3x_3 = -2$, $x_2 - x_3 = 5$, and $0 = 0$. Thus,

$$(-2 - 3x_3, 5 + x_3, x_3) \text{ is a solution for any value of } x_3.$$

Footnote on Example 2: The elementary row operations used in this example were

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \sim \begin{bmatrix} \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} \end{bmatrix} \sim \begin{bmatrix} \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} \end{bmatrix} \sim \begin{bmatrix} \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} \end{bmatrix}$$

Example 3. The following 3-by-3 linear system is inconsistent:

$$\begin{array}{rrrcr} & 3x_2 & -6x_3 & = & 8 \\ x_1 & -2x_2 & +3x_3 & = & -1 \\ 5x_1 & -7x_2 & +9x_3 & = & 0 \end{array}$$

To demonstrate this, consider the sequence of equivalent augmented matrices:

$$\left[\begin{array}{rrr|r} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 5 & -7 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

The last matrix corresponds to $x_1 - 2x_2 + 3x_3 = -1$, $3x_2 - 6x_3 = 8$, and $0 = -3$.

Since $0 \neq -3$, the system has no solution.

Footnote on Example 3: The elementary row operations used in this example were

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \sim \begin{bmatrix} \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \end{bmatrix} \sim \begin{bmatrix} \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \end{bmatrix} \sim \begin{bmatrix} \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \end{bmatrix}$$

Example 4. The following 3-by-4 system has an infinite number of solutions:

$$\begin{array}{rrrrcr} x_1 & +\,x_2 & -\,x_3 & & = & 9 \\ x_1 & +2x_2 & -3x_3 & -4x_4 & = & 9 \\ -3x_1 & -2x_x & +2x_3 & -3x_4 & = & -23 \end{array}$$

To demonstrate this, consider the sequence of equivalent augmented matrices:

$$\left[\begin{array}{rrrr|r} 1 & 1 & -1 & 0 & 9 \\ 1 & 2 & -3 & -4 & 9 \\ -3 & -2 & 2 & -3 & -23 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 0 & 9 \\ 0 & 1 & -2 & -4 & 0 \\ 0 & 1 & -1 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 0 & 9 \\ 0 & 1 & -2 & -4 & 0 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 1 & 0 & 1 & 13 \\ 0 & 1 & 0 & -2 & 8 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 3 & 5 \\ 0 & 1 & 0 & -2 & 8 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right]$$

The last matrix corresponds to $x_1 + 3x_4 = 5$, $x_2 - 2x_4 = 8$, and $x_3 + x_4 = 4$. Thus,

$(5 - 3x_4, 8 + 2x_4, 4 - x_4, x_4)$ is a solution for any value of x_4.

Footnote on Example 4: The elementary row operations used in this example were

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \sim \begin{bmatrix} \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \end{bmatrix} \sim \begin{bmatrix} \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \end{bmatrix} \sim \begin{bmatrix} \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \end{bmatrix} \sim \begin{bmatrix} \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_ \end{bmatrix}$$

Example 5. The following 3-by-2 system is inconsistent:

$$\begin{array}{rrcr} 2x_1 & +4x_2 & = & 26 \\ -3x_1 & -5x_2 & = & -44 \\ -3x_1 & -2x_2 & = & -55 \end{array}$$

To demonstrate this, consider the sequence of equivalent augmented matrices:

$$\left[\begin{array}{rr|r} 2 & 4 & 26 \\ -3 & -5 & -44 \\ -3 & -2 & -55 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 2 & 13 \\ 0 & 1 & -5 \\ 0 & 4 & -16 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 2 & 13 \\ 0 & 1 & -5 \\ 0 & 0 & 4 \end{array}\right]$$

The last matrix corresponds to $x_1 + 2x_2 = 13$, $x_2 = -5$, and $0 = 4$.

Since $0 \neq 4$, the system has no solution.

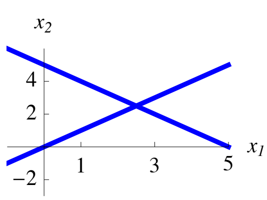
Footnote on Example 5: The elementary row operations used in this example were

$$\left[\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\right] \sim \left[\begin{array}{c} \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \end{array}\right] \sim \left[\begin{array}{c} \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \end{array}\right]$$

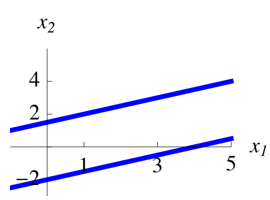
1.1.5 Geometric Interpretations

Linear equations in two variables correspond to lines in the plane. Linear equations in three variables correspond to planes in 3-space. Thus, solution sets for m-by-2 and m-by-3 systems have natural geometric interpretations.

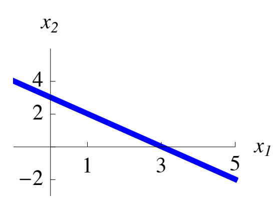
Example 6. Consider the following three 2-by-2 systems:



(1) $\begin{array}{rrcr} x_1 & +x_2 & = & 5 \\ x_1 & -x_2 & = & 0 \end{array}$



(2) $\begin{array}{rrcr} -x_1 & +2x_2 & = & 3 \\ 2x_1 & -4x_2 & = & 8 \end{array}$



(3) $\begin{array}{rrcr} x_1 & +x_2 & = & 3 \\ -2x_1 & -2x_2 & = & -6 \end{array}$

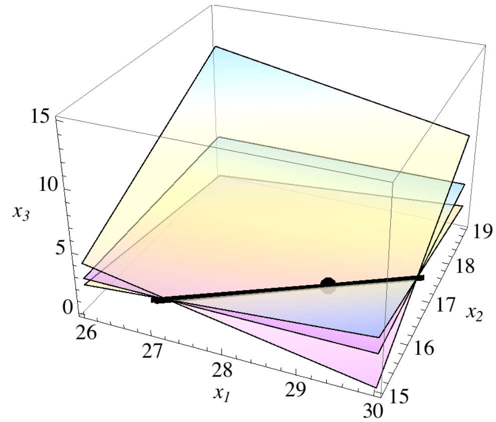
1. System (1) has the unique solution $(2.5, 2.5)$, the point of intersection of the lines.
2. System (2) is inconsistent; the lines are unequal and parallel.
3. System (3) has an infinite number of solutions:

   $$(3 - x_2, x_2) \text{ for any value of } x_2,$$

   since the two equations graph to the same line.

Example 1, revisited. Each pair of planes intersect in a line. The lines are quite close in 3-space.

In addition, the three planes have a single point of intersection, (29, 16, 3).



1.2 Row Reductions and Echelon Forms

1.2.1 Echelon Form and Reduced Echelon Form

A matrix is in *(row) echelon form* when

1. All nonzero rows are above any row of zeros.
2. Each *leading entry* (that is, leftmost nonzero entry) in a row is in a column to the right of the leading entries of the rows above it.
3. All entries in a column below a leading entry are zero.

An echelon matrix is in *reduced form* when (in addition)

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Example 1. The 6-by-11 matrix shown on the left below is in echelon form and the 6-by-11 matrix shown on the right is in reduced echelon form.

$$\begin{bmatrix} 0 & \bullet & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the left matrix, the symbol "$\bullet$" represents a leading nonzero entry and the symbol "$*$" represents any number (either zero or nonzero). In the right matrix, each leading entry has been converted to a 1, and each 1 is the only nonzero entry in its column.

Starting with the augmented matrix of a system of linear equations, the forward phase of the row reduction process will produce a matrix in echelon form. Continuing the process through the backward phase, we get a reduced echelon form. The following theorem explains why reduced echelon form matrices are important.

Theorem (Uniqueness Theorem). Each matrix is row-equivalent (that is, equivalent using elementary row operations) to one and only one reduced echelon form matrix.

Further, we can "read" the solutions to the original system from the reduced echelon form of the augmented matrix.

1.2.2 Pivot Positions, Pivot Columns and Pivots

1. A *pivot position* is a position of a leading entry in an echelon form matrix.
2. A column that contains a pivot position is called a *pivot column*.
3. A *pivot* is a nonzero number that either is used in a pivot position to create zeros or is changed into a leading 1, which in turn is used to create zeros. In general, there is no more than one pivot in any row and no more than one pivot in any column.

Example 1, continued. In the left matrix in Example 1, each "•" is located at a pivot position, and columns 2, 4, 5, 8, and 9 are pivot columns.

Example 2. Consider the following 4-by-5 matrix

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

We first interchange R_1 and R_4:

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Column 1 is a pivot column, and the 1 in the upper left corner is used to change values in rows 2 and 3 to 0 ($R_2 + R_1$ replaces R_2; $R_3 + 2R_1$ replaces R_3):

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Column 2 is a pivot column. We could interchange any of the remaining three rows to identify a pivot, but will just use the 2 in the current second row to change values in rows 3 and 4 to 0 ($R_3 - \frac{5}{2}R_2$ replaces R_3; $R_4 + \frac{3}{2}R_2$ replaces R_4):

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Column 3 is not a pivot column. Column 4 is a pivot column, revealed after we we interchange R_3 and R_4:

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in echelon form. Columns 1, 2, and 4 are pivot columns.

Working with rows 3 and 2 (in that order) to "clear" columns 4 and 2, we get the following reduced echelon form matrix:

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that if the original matrix was the augmented matrix of a 4-by-4 system of linear equations in x_1, x_2, x_3, x_4, then the last matrix corresponds to the equivalent linear system $x_1 - 3x_3 = 5$, $x_2 + 2x_3 = -3$, $x_4 = 0$, $0 = 0$. Thus,

$$(5 + 3x_3, -3 - 2x_3, x_3, 0) \text{ is a solution for any value of } x_3.$$

Footnote on Example 2: The last elementary row operations were:

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} \sim \begin{bmatrix} \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \end{bmatrix} \sim \begin{bmatrix} \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \\ \underline{\hspace{4cm}} \end{bmatrix}$$

Example 3. Consider the following 3-by-6 matrix:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

We first interchange R_1 and R_3:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Column 1 is a pivot column and we use the 3 in the upper left corner to convert the 3 in row 2 to 0 ($R_2 - R_1$ replaces R_2). In addition, R_1 is replaced by $\frac{1}{3}R_1$ for simplicity:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Column 2 is a pivot column. We could use the current row 2, or interchange the second and third rows and use the new row two for pivot. For simplicity, we use the 2 in the second row to convert the 3 in the third row to a zero ($R_3 - \frac{3}{2}R_2$ replaces R_3). In addition, R_2 is replaced by $\frac{1}{2}R_2$ for simplicity:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This matrix is in echelon form; columns 1, 2, and 5 are pivot columns.

Working with rows 3 and 2 (in that order) to "clear" columns 5 and 2, we get the following reduced echelon form matrix:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Note that if the original matrix was the augmented matrix of a 3-by-5 system of linear equations in x_1, x_2, x_3, x_4, x_5, then the last matrix corresponds to the equivalent linear system $x_1 - 2x_3 + 3x_4 = -24$, $x_2 - 2x_3 + 2x_4 = -7$, $x_5 = 4$. Thus,

$$(-24 + 2x_3 - 3x_4, -7 + 2x_3 - 2x_4, x_3, x_4, 4) \text{ is a solution for any values of } x_3,\ x_4.$$

Footnote on Example 3: The last elementary row operations were:

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \sim \begin{bmatrix} \rule{5cm}{0.4pt} \\ \rule{5cm}{0.4pt} \\ \rule{5cm}{0.4pt} \end{bmatrix} \sim \begin{bmatrix} \rule{5cm}{0.4pt} \\ \rule{5cm}{0.4pt} \\ \rule{5cm}{0.4pt} \end{bmatrix}$$

1.2.3 Basic and Free Variables; Solutions to Linear Systems

When solving linear systems,

1. *Basic and Free Variables:*

   A *basic variable* is any variable that corresponds to a pivot column in the augmented matrix of the system, and a *free variable* is any non-basic variable.

2. *Reduced Echelon Forms:*

   From the equivalent system produced using the reduced echelon form, we solve each equation for the basic variable in terms of the free variables (if any) in the equation.

3. *Unique Solution or Infinite Number of Solutions:*

   If there are no free variables, then the original system has a unique solution. If there is at least one free variable, then the original system has an infinite number of solutions.

4. *No Solutions:*

   If an echelon form of the augmented matrix has a row of the form

   $$\begin{bmatrix} 0 & 0 & \cdots & 0 & b \end{bmatrix}$$

   where $b \neq 0$, then the system is inconsistent. (There is no need to continue to find the reduced echelon form.)

Example 2, continued. For the 4-by-4 system in Example 2,

- the basic variables are ____________________________, and
- the free variables are ____________________________.

Example 3, continued. For the 4-by-5 system in Example 3,

- the basic variables are ____________________________, and
- the free variables are ____________________________.

1.3 Vectors and Vector Equations

In linear algebra, we let $\mathbf{R}^m$ be the set of $m \times 1$ matrices of real numbers,

$$\mathbf{R}^m = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} : v_i \in \mathbf{R} \right\}, \text{ and we let } \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \in \mathbf{R}^m \text{ be a specific element. Further,}$$

1. $\boldsymbol{v}$ is known as a *column vector* or simply a *vector*.
2. The value v_i is the i^{th} *component* of $\boldsymbol{v}$.
3. The *zero vector*, $\boldsymbol{O}$, is the vector all of whose components equal zero.

1.3.1 Vector Sum, Scalar Product, and Linear Combinations of Vectors

Suppose $\boldsymbol{v}, \boldsymbol{w} \in \mathbf{R}^m$ and $c, d \in \mathbf{R}$.

1. The *vector sum* $\boldsymbol{v} + \boldsymbol{w}$ is the vector obtained by componentwise addition. That is, $\boldsymbol{v} + \boldsymbol{w}$ is the vector whose i^{th} component is $v_i + w_i$ for each i.
2. The *scalar product* $c\boldsymbol{v}$ is the vector obtained by multiplying each component by c. That is, $c\boldsymbol{v}$ is the vector whose i^{th} component is cv_i for each i.
3. The vector $c\boldsymbol{v} + d\boldsymbol{w}$ is known as a *linear combination* of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$.

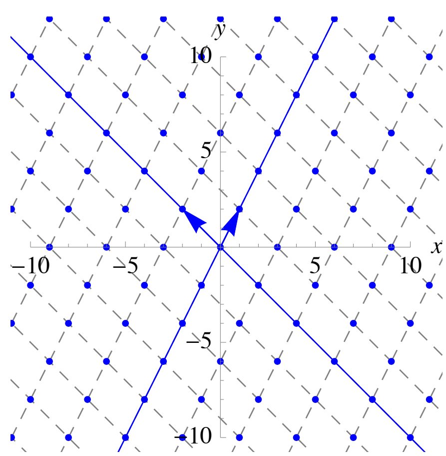
Example in 2-space. Let $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ be vectors in $\mathbf{R}^2$. These vectors can be represented as points in the plane or as directed line segments whose initial point is the origin. In the plot,

- Vector $\boldsymbol{v_1}$ is represented as the directed line segment from $(0,0)$ to $(1,2)$, and points on $y = 2x$ represent scalar multiples of $\boldsymbol{v_1}$:

$$c\boldsymbol{v_1}, \text{ where } c \in \mathbf{R}.$$

- Vector $\boldsymbol{v_2}$ is represented as the directed line segment from $(0,0)$ to $(-2,2)$, and points on $y = -x$ represent scalar multiples of $\boldsymbol{v_2}$:

$$d\boldsymbol{v_2}, \text{ where } d \in \mathbf{R}.$$



- Linear combinations of the two vectors, $c\boldsymbol{v_1} + d\boldsymbol{v_2}$, are in one-to-one correspondence with $\mathbf{R}^2$. The plot shows a "grid" of linear combinations where either c or d is an integer.

Problem. Write $\boldsymbol{w} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ as a linear combination of $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

Example in 3-space. Let $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ be vectors in $\mathbf{R}^3$.

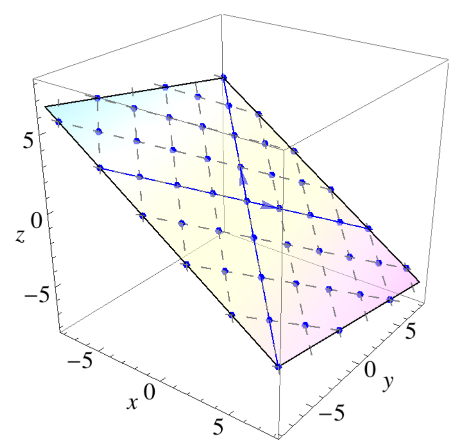
- The solid lines represent scalar multiples of the two vectors:

$$c\boldsymbol{v_1} \text{ for } c \in \mathbf{R}, \text{ and } d\boldsymbol{v_2} \text{ for } d \in \mathbf{R}.$$

- The plane shown represents the linear combinations of the two vectors:

$$c\boldsymbol{v_1} + d\boldsymbol{v_2}, \text{ for } c, d \in \mathbf{R}.$$

- The "grid" represents the linear combinations where either c or d is an integer.



Problem. Let $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, as above.

(a) Suppose that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\boldsymbol{v_1} + d\boldsymbol{v_2}$ for some $c, d \in \mathbf{R}$. Find an equation relating x, y, z.

(b) Can $\boldsymbol{w} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ be written as a linear combination of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$? Why?

1.3.2 Linear Combinations and Spans

If $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n} \in \mathbf{R}^m$ and $c_1, c_2, \ldots, c_n \in \mathbf{R}$, then

$\boldsymbol{w} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n}$ is a *linear combination* of the vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, ..., $\boldsymbol{v_n}$.

The *span* of the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ is the collection of all linear combinations of $\boldsymbol{v_1} \ldots \boldsymbol{v_n}$:

$$\text{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\} = \{c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n} \ : \ c_i \in \mathbf{R} \text{ for each } i\} \subseteq \mathbf{R}^m.$$

<u>*Note:*</u> The span of a set of vectors always contains the zero vector $\boldsymbol{O}$ (let each $c_i = 0$).

For example,

1. $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \left\{c\begin{bmatrix} 1 \\ 2 \end{bmatrix} \ : \ c \in \mathbf{R}\right\}$ corresponds to the line $y = 2x$ in $\mathbf{R}^2$.

2. $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right\} = \left\{c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} -2 \\ 2 \end{bmatrix} \ : \ c, d \in \mathbf{R}\right\} = \mathbf{R}^2$. That is, every vector in $\mathbf{R}^2$ can be written as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

3. $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \end{bmatrix}\right\} = \left\{c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} -2 \\ 2 \end{bmatrix} + e\begin{bmatrix} 7 \\ 5 \end{bmatrix} \ : \ c, d, e \in \mathbf{R}\right\} = \mathbf{R}^2$.
   That is, every vector in $\mathbf{R}^2$ can be written as a linear combination of the three listed vectors. (In fact, the first two vectors were already sufficient to represent each $\boldsymbol{v} \in \mathbf{R}^2$.)

4. $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right\} = \left\{c\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + d\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \ : \ c, d \in \mathbf{R}\right\}$ corresponds to the

   plane with equation ______________________________ in $\mathbf{R}^3$.

5. $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right\}$ since the third vector in the list is a linear combination of the first two.

Exercise. Let $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\boldsymbol{v_2} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v_3} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$. Demonstrate that

$$\mathrm{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} = \{\mathrm{c}\boldsymbol{v_1} + \mathrm{d}\boldsymbol{v_2} + \mathrm{e}\boldsymbol{v_3} \ : \ \mathrm{c}, \mathrm{d}, \mathrm{e} \in \mathbf{R}\} = \mathbf{R}^3.$$

That is, demonstrate that every vector in $\mathbf{R}^3$ can be written as a linear combination of the three vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\boldsymbol{v_3}$.

1.3.3 Vector Equations, Spans and Solution Sets

Consider an m-by-n system of equations in variables x_1, x_2, ..., x_n, where each equation is of the form

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \quad \text{for } i = 1, 2, \ldots, m.$$

Let $\boldsymbol{a_j}$ be the vector of coefficients of x_j, and $\boldsymbol{b}$ be the vector of right hand side values.

The m-by-n system can be rewritten as a *vector equation*:

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \boldsymbol{b}.$$

Note: The system of equations is consistent if and only if $\boldsymbol{b} \in \text{Span}\{\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}\}$.

For example, the linear system on the left below becomes the vector equation on the right:

$$\begin{array}{rrrcr} x_1 & +5x_2 & -3x_3 & = & -4 \\ -x_1 & -4x_2 & +\ x_3 & = & 3 \\ -2x_1 & -7x_2 & & = & 12 \end{array} \quad \Rightarrow \quad x_1 \begin{bmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix} + x_2 \begin{bmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix} + x_3 \begin{bmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix} = \begin{bmatrix} \_\_\_ \\ \_\_\_ \\ \_\_\_ \end{bmatrix}.$$

Problem. Starting with the vector equation $x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + x_3\boldsymbol{a_3} = \boldsymbol{b}$ shown on the right above,

(a) Determine if the system is consistent. That is, determine if $\boldsymbol{b} \in \text{Span}\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$.

(b) Find the reduced echelon form of the matrix whose columns are the $\boldsymbol{a_i}$'s:

$$A = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \end{bmatrix}.$$

(c) What do we know about the set $\text{Span}\{\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}\}$? Be as complete as possible.

1.4 Matrix Equations and Solution Sets

As above, an m-by-n system of linear equations,

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \quad \text{for } i = 1, 2, \ldots, m,$$

leads to a vector equation,

$$x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots + x_n\boldsymbol{a_n} = \boldsymbol{b},$$

where $\boldsymbol{a_j}$ is the vector of coefficients of x_j, and $\boldsymbol{b}$ is the vector of right hand side values.

1.4.1 Coefficient Matrix and Matrix-Vector Product

1. *Coefficient Matrix:*

   The *coefficient matrix* A of an m-by-n system of linear equations is the $m \times n$ matrix whose columns are the $\boldsymbol{a_j}$'s:

   $$A = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \cdots & \boldsymbol{a_n} \end{bmatrix}.$$

2. *Matrix-Vector Product:*

   If A is an $m \times n$ matrix and $\boldsymbol{v} \in \mathbf{R}^n$, then the *matrix-vector product of A with $\boldsymbol{v}$* is defined to be the the following linear combination of the columns of A:

   $$A\boldsymbol{v} = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \cdots & \boldsymbol{a_n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\boldsymbol{a_1} + v_2\boldsymbol{a_2} + \cdots + v_n\boldsymbol{a_n}.$$

For example, if $A = \begin{bmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, then

$$A\boldsymbol{v} = 3\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - 1\begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}.$$

Row-column definition. The definition of the matrix-vector product as a linear combination of column vectors is equivalent to the usual "row-column definition" of the product of two matrices. For example,

$$A\boldsymbol{v} = \begin{bmatrix} 1 & 5 & -3 \\ \hline -1 & -4 & 1 \\ \hline -2 & -7 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(3) + 5(-1) - 3(2) \\ -1(3) - 4(-1) + 1(2) \\ -2(3) - 7(-1) + 0(2) \end{bmatrix} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}.$$

Linearity Property of Matrix-Vector Products:

The matrix-vector product of a linear combination of k vectors in $\mathbf{R}^n$ is equal to that linear combination of matrix-vector products:

$$A\left(c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}\right) = c_1A\boldsymbol{v_1} + c_2A\boldsymbol{v_2} + \cdots + c_kA\boldsymbol{v_k} \quad \text{for all } c_i \in \mathbf{R},\ \boldsymbol{v_i} \in \mathbf{R}^n.$$

Another way of saying this is

"Multiplication by A distributes over addition, and we can factor out constants."

1.4.2 Matrix Equation, Span and Consistency

Let $A = \begin{bmatrix}\boldsymbol{a_1} & \boldsymbol{a_2} & \cdots & \boldsymbol{a_n}\end{bmatrix}$ be the coefficient matrix of an m-by-n system of linear equations.

If $\boldsymbol{x} \in \mathbf{R}^n$ a vector of unknowns and $\boldsymbol{b} \in \mathbf{R}^m$ is the vector of right hand side values, then

$$A\boldsymbol{x} = \boldsymbol{b} \quad \text{is known as the } \textit{matrix equation} \text{ of the system.}$$

For example, the following 3-by-3 system of linear equations,

$$\begin{array}{rrrcr} x_1 & +5x_2 & -3x_3 & = & -4 \\ -x_1 & -4x_2 & +\,x_3 & = & 3 \\ -2x_1 & -7x_2 & & = & 12 \end{array}$$

corresponds to the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} -4 \\ 3 \\ 12 \end{bmatrix}.$$

The following theorem gives criteria for determining when a matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for all possible right hand side vectors $\boldsymbol{b}$.

Theorem (Consistency Theorem). Let $A = \begin{bmatrix}\boldsymbol{a_1} & \boldsymbol{a_2} & \cdots & \boldsymbol{a_n}\end{bmatrix}$ be an $m \times n$ coefficient matrix, and $\boldsymbol{x} \in \mathbf{R}^n$ be a vector of unknowns.

The following statements are equivalent:

(a) The equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for all $\boldsymbol{b} \in \mathbf{R}^m$.

(b) Each $\boldsymbol{b} \in \mathbf{R}^m$ is a linear combination of the coefficient vectors $\boldsymbol{a_j}$.

(c) The columns of the matrix A span $\mathbf{R}^m$. That is, $\text{Span}\{\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}\} = \mathbf{R}^m$.

(d) A has a pivot position in each row.

Problem 1. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 1 & -10 \end{bmatrix}$.

(1) Determine if $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for all $\boldsymbol{b} \in \mathbf{R}^3$.

(2) If it is not consistent for all $\boldsymbol{b}$, find an equation characterizing when $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

Problem 2. Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 11 \end{bmatrix}$.

(1) Determine if $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for all $\boldsymbol{b} \in \mathbf{R}^3$.

(2) If it is not consistent for all $\boldsymbol{b}$, find an equation characterizing when $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

Problem 3. A matrix can have at most 1 pivot position in any row or any column.

What, if anything, does this fact tell you about determining consistency for all $\boldsymbol{b}$ when $m < n$, $m = n$, $m > n$? Be as complete as possible in your explanation.

1.4.3 Solution Sets and Homogeneous Systems

Let $A\boldsymbol{x} = \boldsymbol{b}$ be the matrix equation of an m-by-n system of linear equations.

1. *Solution Set:*

   The *solution set* of the system is $\{\boldsymbol{x} \ : \ A\boldsymbol{x} = \boldsymbol{b}\} \subseteq \mathbf{R}^n$. A solution set can be empty, contain one element or contain an infinite number of elements.

2. *Homogeneous System:*

   If $\boldsymbol{b} = \boldsymbol{O}$ (the zero vector), then the system is said to be *homogeneous*.

Homogeneous systems satisfy the following properties:

1. Every homogeneous system is consistent since $\boldsymbol{x} = \boldsymbol{O}$ is a solution to the system.
2. If $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}$ are solutions to the homogeneous system and $c_1, c_2, \ldots, c_k$ are constants, then the linear combination

   $$\boldsymbol{w} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} \text{ is also a solution to the system.}$$

3. The solution set of $A\boldsymbol{x} = \boldsymbol{O}$ can be written as the span of a certain set of vectors.

Note: The solution "$\boldsymbol{x} = \boldsymbol{O}$" is called the *trivial solution* to the homogeneous system.

Problem 4. Demonstrate the second property of homogeneous systems.

Writing the solution set of $A\boldsymbol{x}=\boldsymbol{O}$ as a span. The following example illustrates the technique for writing the solution set of a homogeneous system as a span:

$$\begin{array}{rrrcl} x_1 & +5x_2 & -3x_3 & = & 0 \\ -x_1 & -4x_2 & +x_3 & = & 0 \\ -2x_1 & -7x_2 & & = & 0 \end{array} \implies \begin{bmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since

$$\left[\begin{array}{rrr|r} 1 & 5 & -3 & 0 \\ -1 & -4 & 1 & 0 \\ -2 & -7 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 5 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 5 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} x_1 = -7x_3 \\ x_2 = 2x_3 \\ x_3 \text{ free,} \end{array}$$

we can write the solutions to the homogeneous system as vectors satisfying

$$\boldsymbol{x} = \begin{bmatrix} -7x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} \text{ where } x_3 \text{ is free. Thus, Span}\left\{ \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} \right\} = \left\{ c \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} : c \in \mathbf{R} \right\}$$

is the solution set of the homogeneous system above.

Problem 5. In each case, write the solution set of $A\boldsymbol{x} = \boldsymbol{O}$ as a span of a set of vectors.

(a) Let $A = \begin{bmatrix} 2 & -2 \\ 1 & 2 \\ 1 & 8 \end{bmatrix}$.

(b) Let $A = \begin{bmatrix} 1 & 2 & -8 \\ 2 & 4 & -12 \end{bmatrix}$.

(c) Let $A = \begin{bmatrix} 2 & 4 & -6 & 8 \\ 4 & 8 & 4 & 0 \\ 3 & 6 & -1 & 4 \end{bmatrix}$.

Writing the solution set of $\boldsymbol{Ax}=\boldsymbol{O}$ in parametric vector form. If the solution set to $A\boldsymbol{x} = \boldsymbol{O}$ is $\text{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$, then the solutions can be written in *parametric vector form* as follows:

$$\boldsymbol{x} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \ldots + c_k\boldsymbol{v_k}, \text{ where } c_1, c_2, \ldots, c_k \in \mathbf{R},$$

and the c_i's are the *parameters* in the general equation for the solution.

For example, the parametric vector form for the solutions in Problem 5(b) is

$$\boldsymbol{x} = c \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}, \quad \text{for } c \in \mathbf{R} \text{ (only one parameter is needed);}$$

and the parametric vector form for the solutions in Problem 5(c) is

$$\boldsymbol{x} = c_1 \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} + c_2 \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}, \quad \text{for } c_1, c_2 \in \mathbf{R} \text{ (two parameters are needed).}$$

1.4.4 Solution Sets and Nonhomogeneous Systems

A linear system is *nonhomogeneous* if it is not homogeneous. That is, a linear system is nonhomogeneous if its matrix equation is of the form $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b} \neq \boldsymbol{O}$.

> ***Theorem (Solution Sets).*** Suppose the equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent and let $\boldsymbol{p}$ be a particular solution.
>
> Then the solution set of $A\boldsymbol{x} = \boldsymbol{b}$ is the set of all vectors of the form
>
> $$\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{v_h}, \text{ where } \boldsymbol{v_h} \text{ is a solution to } A\boldsymbol{x} = \boldsymbol{O}.$$
>
> That is, each solution can be written as the vector sum of a particular solution to the nonhomogeneous system and a solution to the homogeneous system with the same coefficient matrix.

For example, let $A = \begin{bmatrix} 2 & 4 & -6 \\ 4 & 8 & -10 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$. Since

$$[\ A \mid \boldsymbol{b}\ ] = \left[\begin{array}{ccc|c} 2 & 4 & -6 & 0 \\ 4 & 8 & -10 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \Rightarrow \quad \begin{array}{c} x_1 = 6 - 2x_2 \\ x_2 \text{ is free} \\ x_3 = 2 \end{array}$$

$$\Rightarrow\ \boldsymbol{x} = \begin{bmatrix} 6 - 2x_2 \\ x_2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \text{ where } x_2 \text{ is free,}$$

we can write $\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{v_h}$, where $\boldsymbol{p} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}$ and $\boldsymbol{v_h} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ for some x_2.

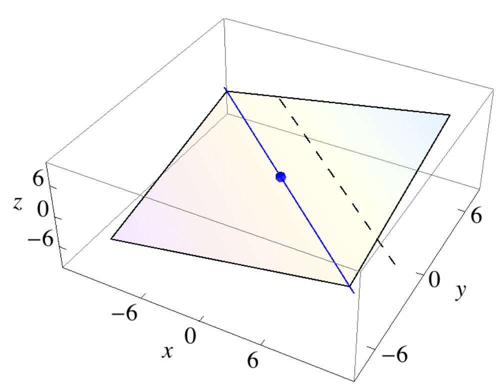
We can visualize the solution sets to both systems as follows:

- <u>*Solid Line:*</u> The solution to the homogeneous system is the solid line

$$c \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \text{ for some } c.$$

- <u>*Dashed Line:*</u> The solution to the original nonhomogeneous system is the dashed line

$$\begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \text{ for some } c.$$



(The solutions are written in parametric vector form, using c as the parameter.)

Each point on the dashed line is obtained by adding $\boldsymbol{p}$ to a point on the solid line. The lines are parallel and lie on the plane $x + 2y - 3z = 0$.

Note: Solution Set is the Shift of a Span

Since the set of solutions to $A\boldsymbol{x} = \boldsymbol{O}$ is the span of a set of vectors, the set of solutions to $A\boldsymbol{x} = \boldsymbol{b}$ can be described as the shift of a span, where each solution of the homogeneous system is *shifted* by the particular solution $\boldsymbol{p}$.

Problem 6. Let $A = \begin{bmatrix} 2 & 4 & -6 & 8 \\ 4 & 8 & 4 & 0 \\ 3 & 6 & -1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 18 \\ 4 \\ 11 \end{bmatrix}$.

Write the solutions to $A\boldsymbol{x} = \boldsymbol{b}$ in the form $\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{v_h}$, where $\boldsymbol{p}$ is a particular solution to the nonhomogeneous system and $\boldsymbol{v_h}$ is a general solution to the homogeneous system $A\boldsymbol{x} = \boldsymbol{O}$.

1.5 Linearly Independent and Linearly Dependent Sets

The set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\} \subseteq \mathbf{R}^m$ is said to be *linearly independent* when

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n} = \boldsymbol{O} \text{ only when each } c_j = 0.$$

Otherwise, the set is said to be *linearly dependent.*

<u>*Note:*</u> If $A = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \cdots & \boldsymbol{v_n} \end{bmatrix}$ is the matrix whose columns are the $\boldsymbol{v_j}$'s, then

$$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\} \text{ is linearly independent} \Leftrightarrow A\boldsymbol{x} = \boldsymbol{O} \text{ has the trivial solution only.}$$

Problem 1. In each case: (1) Determine if the set $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ is linearly independent or not; and (2) If the set is linearly dependent, then find a linear dependence relationship among the vectors (that is, find constants c_j not all equal to zero satisfying $c_1\boldsymbol{v_1} + \ldots + c_n\boldsymbol{v_n} = \boldsymbol{O}$).

(*a*) Let $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbf{R}^4.$

(b) Let $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} = \left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 10 \end{bmatrix} \right\} \subseteq \mathbf{R}^3.$

(c) Let $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \end{bmatrix} \right\} \subseteq \mathbf{R}^2.$

The following theorem gives an important property of linearly independent sets.

Theorem (Uniqueness Theorem). If $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\} \subseteq \mathbf{R}^m$ is a linearly independent set and $\boldsymbol{w} \in \mathrm{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$, then $\boldsymbol{w}$ can be written uniquely as a linear combination of the $\boldsymbol{v_i}$'s. That is, we can write

$$\boldsymbol{w} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n}$$

for a unique ordered list of scalars c_1, c_2, ..., c_n.

Problem 2. Demonstrate that the uniqueness theorem is true.

Facts about linearly independent and linearly dependent sets include the following

1. If $\boldsymbol{v_i} = \boldsymbol{O}$ for some i, then the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ is linearly dependent.
2. Suppose that $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are nonzero vectors. Then the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ is linearly independent if and only if $\boldsymbol{v_2} \neq c\boldsymbol{v_1}$ for some c.
3. If $m < n$, then the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ is linearly dependent.
4. Suppose $n \geq 2$ and each $\boldsymbol{v_i} \neq \boldsymbol{O}$. Then, the set is linearly dependent if and only if one vector in the set can be written as a linear combination of the others.
5. Let A be the $m \times n$ matrix whose columns are the $\boldsymbol{v_j}$'s: $A = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \cdots & \boldsymbol{v_n} \end{bmatrix}$. Then, the set is linearly independent if and only if the homogeneous system $A\boldsymbol{x} = \boldsymbol{O}$ has the trivial solution only.
6. Let A be the $m \times n$ matrix whose columns are the $\boldsymbol{v_j}$'s: $A = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \cdots & \boldsymbol{v_n} \end{bmatrix}$. Then, the set is linearly independent if and only if A has a pivot in every column.

Problem 3. Which, if any, of the following sets are linearly independent? Why?

(a) $$\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 12 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right\}.$$

(b) $$\left\{ \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} \right\}.$$

(c) $$\left\{ \begin{bmatrix} 1 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -18 \\ 6 \end{bmatrix} \right\}.$$

(d) $$\left\{ \begin{bmatrix} 7 \\ 4 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

1.6 Linear and Matrix Transformations

1.6.1 Transformations, Images and Pre-Images

1. *Transformation:* A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is a function (or rule) that assigns to each $\boldsymbol{x} \in \mathbf{R}^n$ a value $T(\boldsymbol{x}) = \boldsymbol{b} \in \mathbf{R}^m$. Thus,

   - *Domain:*
     The domain of T is all of n-space: $\text{Domain}(T) = \mathbf{R}^n$.
   - *Range:*
     The range of T is a subset of m-space: $\text{Range}(T) = \{T(\boldsymbol{x}) \ : \ \boldsymbol{x} \in \mathbf{R}^n\} \subseteq \mathbf{R}^m$.

2. *Image:*

   The value $T(\boldsymbol{x})$ for a given $\boldsymbol{x}$ is also called its *image*.

3. *Pre-Image:*

   Each $\boldsymbol{x} \in \mathbf{R}^n$ whose image is $\boldsymbol{b} \in \text{Range}(T)$ is said to be a *pre-image* of $\boldsymbol{b}$.

Note that images are unique, but pre-images may not be unique.

Problem 1. In each case, find the range of the transformation. That is, find Range(T).

(a) $T : \mathbf{R}^2 \to \mathbf{R}^2$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_2 \end{bmatrix}$.

(b) $T : \mathbf{R}^2 \to \mathbf{R}^2$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + 6x_2 \\ x_1 - 3x_2 \end{bmatrix}$.

(c) $T : \mathbf{R}^4 \to \mathbf{R}^3$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_3 + x_4 \\ x_2 + x_4 \\ x_1 + x_2 + x_3 + x_4 \end{bmatrix}$.

(d) $T : \mathbf{R}^3 \to \mathbf{R}^3$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{bmatrix}$.

1.6.2 One-To-One and Onto Transformations

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a transformation.

1. T is said to be <u>*one-to-one*</u> if each $\boldsymbol{b} \in \mathbf{R}^m$ is the image of at most one $\boldsymbol{x} \in \mathbf{R}^n$.
2. T is said to be <u>*onto*</u> if $\text{Range}(T) = \mathbf{R}^m$.

Problem 1(a), continued. Is $T : \mathbf{R}^2 \to \mathbf{R}^2$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_2 \end{bmatrix}$ one-to-one and/or onto? Explain.

Problem 1(b), continued. Is $T : \mathbf{R}^2 \to \mathbf{R}^2$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + 6x_2 \\ x_1 - 3x_2 \end{bmatrix}$ one-to-one and/or onto? Explain.

Problem 1(c), continued. Is $T : \mathbf{R}^4 \to \mathbf{R}^3$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_3 + x_4 \\ x_2 + x_4 \\ x_1 + x_2 + x_3 + x_4 \end{bmatrix}$ one-to-one and/or onto? Explain.

Problem 1(d), continued. Is $T : \mathbf{R}^3 \to \mathbf{R}^3$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{bmatrix}$ one-to-one and/or onto? Explain.

1.6.3 Linear Transformations

$T : \mathbf{R}^n \to \mathbf{R}^m$ is said to be a *linear transformation* if the following two conditions are satisfied:

1. $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ for $c \in \mathbf{R}$ and $\boldsymbol{v} \in \mathbf{R}^n$, and
2. $T(\boldsymbol{v} + \boldsymbol{w}) = T(\boldsymbol{v}) + T(\boldsymbol{w})$ for $\boldsymbol{v}, \boldsymbol{w} \in \mathbf{R}^n$.

The conditions for linear transformations lead to the following facts:

Fact 1: If T is a linear transformation and $\boldsymbol{w}$ is a linear combination of $\boldsymbol{v_1}$,$\boldsymbol{v_2}$, ..., $\boldsymbol{v_k}$, then $T(\boldsymbol{w})$ is a linear combination of $T(\boldsymbol{v_1})$,$T(\boldsymbol{v_2})$, ..., $T(\boldsymbol{v_k})$. In symbols,

$$\boldsymbol{w} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} \quad \Longrightarrow \quad T(\boldsymbol{w}) = c_1T(\boldsymbol{v_1}) + c_2T(\boldsymbol{v_2}) + \cdots + c_kT(\boldsymbol{v_k}).$$

Fact 2: If T is a linear transformation, then T maps the span of a set of vectors to the span of the set of images of the vectors. In symbols,

$$T\Big(\text{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}\Big) \;=\; \text{Span}\{T(\boldsymbol{v_1}), T(\boldsymbol{v_2}), \ldots, T(\boldsymbol{v_k})\}.$$

Fact 3: If $T(\boldsymbol{x}) = A\boldsymbol{x}$, for some $m \times n$ matrix A, then T is a linear transformation.

Problem 2. Which, if any, transformations from Problem 1 are linear transformations? Explain.

Problem 3. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation, and suppose that

$$T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} -3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(*a*) Find $T\left(\begin{bmatrix} 2 \\ -6 \end{bmatrix}\right)$.

(*b*) Find a general formula for $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

1.6.4 Linear and Matrix Transformations

The following theorem says that every linear transformation can be written as a matrix transformation. Formally,

> ***Matrix Transformation Theorem.*** If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then the rule for T can be written as $T(\boldsymbol{x}) = A\boldsymbol{x}$ for a unique $m \times n$ matrix A.

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation, and let $\boldsymbol{e_j} \in \mathbf{R}^n$ be the vector whose j^{th} component is 1 and whose remaining components are 0, for $j = 1, 2, \ldots, n$. Then

1. Each $\boldsymbol{x} \in \mathbf{R}^n$ can be written uniquely as a linear combination of the $\boldsymbol{e_j}$'s:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = x_1\boldsymbol{e_1} + x_2\boldsymbol{e_2} + \ldots + x_n\boldsymbol{e_n}.$$

2. Since T is a linear transformation,

$$\begin{aligned} T(\boldsymbol{x}) &= x_1T(\boldsymbol{e_1}) + x_2T(\boldsymbol{e_2}) + \ldots + x_nT(\boldsymbol{e_n}) \\ &= \begin{bmatrix} T(\boldsymbol{e_1}) & T(\boldsymbol{e_2}) & \cdots & T(\boldsymbol{e_n}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\boldsymbol{x}. \end{aligned}$$

3. Thus, the rule for T corresponds to matrix multiplication, where A is the matrix whose columns are the images of the $\boldsymbol{e_j}$'s: $A = \begin{bmatrix} T(\boldsymbol{e_1}) & T(\boldsymbol{e_2}) & \cdots & T(\boldsymbol{e_n}) \end{bmatrix}$.

> *Note: Standard Matrix and Standard Basis Vectors*
>
> The matrix A of the linear transformation is known as the *standard matrix* of T.
>
> The vectors $\boldsymbol{e_1}$, $\boldsymbol{e_2}$, ..., $\boldsymbol{e_n}$ are known as the *standard basis vectors* of $\mathbf{R}^n$.

If we know the images of the standard basis vectors under the linear transformation T, then we know the standard matrix of T (and the formula for T in terms of the standard matrix).

For example, if $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation,

$$T(\boldsymbol{e_1}) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \text{ and } T(\boldsymbol{e_2}) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \text{ then } A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & 1 \end{bmatrix}.$$

Problem 4. Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation,

$$T(\boldsymbol{e_1} + 2\boldsymbol{e_2}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } T(\boldsymbol{e_1} - \boldsymbol{e_2}) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$$

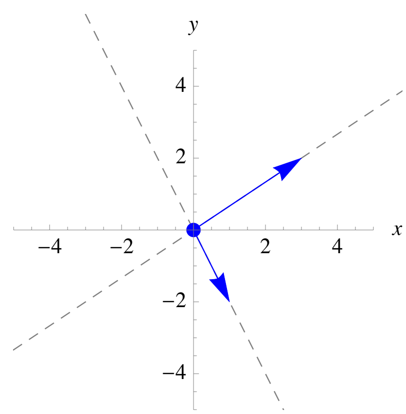
Find $T(\boldsymbol{e_1})$, $T(\boldsymbol{e_2})$ and the standard matrix of T.

Images of lines in the plane. If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation, then T maps lines to either lines or points.

Problem 5. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation with rule

$$T(\boldsymbol{x}) = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \boldsymbol{x}.$$

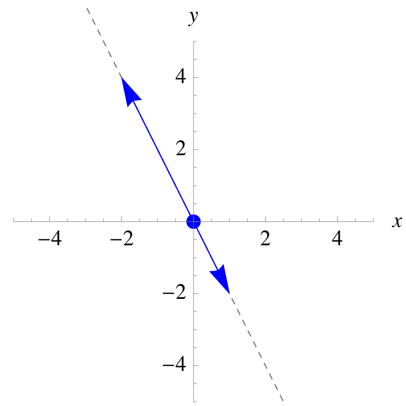
Describe the image of the line $y = 2x$.



Problem 6. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation with rule

$$T(\boldsymbol{x}) = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \boldsymbol{x}.$$

Describe the image of the line $y = \frac{x}{2}$ (or $x = 2y$).



One-to-one and onto transformations, revisited. Recall that $T : \mathbf{R}^n \to \mathbf{R}^m$ is

- *one-to-one* if each $\boldsymbol{b} \in \text{Range}(T)$ is the image of at most one $\boldsymbol{x} \in \mathbf{R}^n$, and
- *onto* if $\text{Range}(T) = \mathbf{R}^m$.

The following theorem relates these definitions to properties of the standard matrix of a linear transformation.

> ***Theorem (Linear Transformations).*** Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be the linear transformation with rule $T(\boldsymbol{x}) = A\boldsymbol{x}$, where A is an $m \times n$ matrix. Then
>
> 1. T is onto if and only if the columns of A span $\mathbf{R}^m$.
> 2. T is one-to-one if and only if the columns of A are a linearly independent set.

Aside 1: T is onto iff A has a pivot ______________________________________.

Aside 2: T is one-to-one iff A has a pivot ______________________________________.

Problem 7. Which, if any, of the following transformations are one-to-one? Which, if any, are onto? Why?

(*a*) $T : \mathbf{R}^2 \to \mathbf{R}^3$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ 3x_1 - x_2 \\ 3x_1 - 2x_2 \end{bmatrix}$

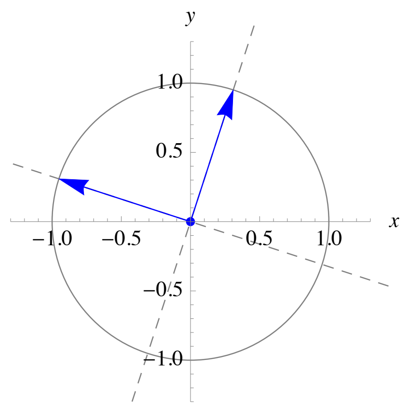
(*b*) $T : \mathbf{R}^4 \to \mathbf{R}^3$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 4x_2 + 8x_3 + x_4 \\ 2x_2 - 8x_3 + 3x_4 \\ 5x_4 \end{bmatrix}$

(*c*) $T : \mathbf{R}^3 \to \mathbf{R}^3$ with rule $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 + x_3 \\ 3x_1 - 2x_2 + 4x_3 \end{bmatrix}$

Footnote: Rotations in 2-space. Recall that positive angles are measured counterclockwise from the positive x-axis, and negative angles are measured clockwise from the positive x-axis.

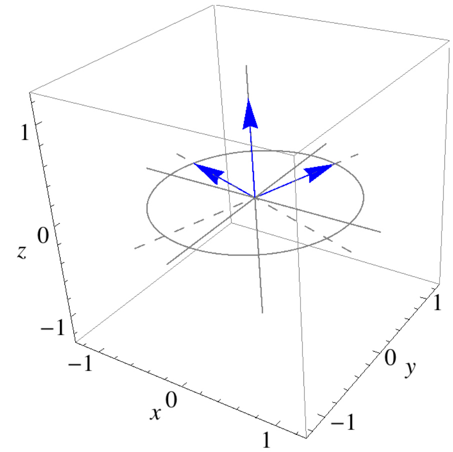
The transformation that rotates points in the plane at an angle θ is a linear transformation.

For example, the plot on the right shows the images of the standard basis vectors under the rotation through the positive angle $\theta = 2\pi/5$.



Problem 8. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation about the origin through angle θ. Find the standard matrix of T. Your answer should be in terms of $\sin(\theta)$ and $\cos(\theta)$.

Footnote: Rotations about the z-axis. Assume that positive angles are measured counterclockwise from the positive x-axis, and negative angles are measured clockwise from the positive x-axis, when looking down from the positive z-axis.



The transformation that rotates points in 3-space at angle θ around the z-axis is a linear transformation.

For example, the plot on the right shows the images of the standard basis vectors under the rotation through the positive angle $\theta = 2\pi/5$.

Problem 9. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the rotation about the z-axis through angle θ. Find the standard matrix of T. Your answer should be in terms of $\sin(\theta)$ and $\cos(\theta)$.