MATH2210 Notebook 2 — Spring 2018

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This notebook is concerned with matrix methods in linear algebra. The notes correspond to material in Chapters 2 and 3 of the Lay textbook.

2.1 Matrices and Their Operations

2.1.1 Basic Definitions and Conventions

1. *Matrix:*

   A matrix A is a rectangular array with m rows and n columns,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \cdots & \boldsymbol{a_n} \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}.$$

2. *Size of Matrix:*

   The expression "$m \times n$" (read "m-by-n") is the size of the matrix A.

3. *Entries of Matrix:*

   The number in the row i, column j position of the matrix is denoted by a_{ij} and is referred to as the (i, j) entry of A. For convenience, we often write $(A)_{ij} = a_{ij}$.

4. *Columns of Matrix:*

   The matrix A is often written as an array of n columns: $\boldsymbol{a_j}$, $j = 1, 2, \ldots, n$.

   Each column of A is a vector in $\mathbf{R}^m$.

5. *Square Matrix of Order n:*

   If the number of rows equals the number of columns ($m = n$), we say that A is a square matrix of order n. Square matrices are important in applications.

6. *Identity Matrix of Order n:*

   The identity matrix of order n is the square matrix of order n whose columns are the standard basis vectors in $\mathbf{R}^n$. We write

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \cdots & \boldsymbol{e_n} \end{bmatrix},$$

   where $\boldsymbol{e_j}$ is the vector with 1 in position j, and 0 everywhere else, for $j = 1, 2, \ldots, n$.

   We often use "I" without a subscript if the size of the matrix is clear.

2.1.2 Focus on Square Matrices

If A is a square matrix of order n, then the elements

a_{11}, a_{22}, …, a_{nn} are known as the *diagonal elements* of A.

Square matrices are often classified in a way that focuses on the diagonal elements:

1. A is said to be *symmetric* if $a_{ij} = a_{ji}$ for all i and j.
2. A is said to be *upper triangular* $a_{ij} = 0$ when $i > j$.
3. A is said to be *lower triangular* $a_{ij} = 0$ when $i < j$.
4. A is said to be *triangular* if it is either upper triangular or lower triangular.
5. A is said to be *diagonal* if $a_{ij} = 0$ when $i \neq j$.

Problem 1. Consider the following 3×3 matrices

$$A_1 = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, \quad A_2 = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, \quad A_3 = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \quad \text{and} \quad A_4 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

where a through f are any numbers. Classify each matrix in as many ways as possible based on the list of 5 types of square matrices above.

2.1.3 Matrix Sum, Scalar Product, Linear Combination

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices, and let $c, d \in \mathbf{R}$ be scalars. Then

1. *Matrix Sum of Compatible Matrices:*

   The matrix sum of A and B is the $m \times n$ matrix

   $$A + B = \begin{bmatrix} \boldsymbol{a_1} + \boldsymbol{b_1} & \boldsymbol{a_2} + \boldsymbol{b_2} & \cdots & \boldsymbol{a_n} + \boldsymbol{b_n} \end{bmatrix}.$$

   Equivalently, $A + B$ is the matrix with (i, j)-entry $(A + B)_{ij} = a_{ij} + b_{ij}$.

2. *Scalar Product:*

   The scalar product of A by c is the $m \times n$ matrix

   $$cA = \begin{bmatrix} c\boldsymbol{a_1} & c\boldsymbol{a_2} & \cdots & c\boldsymbol{a_n} \end{bmatrix}.$$

   Equivalently, cA is the matrix whose (i, j)-entry is $(cA)_{ij} = ca_{ij}$.

3. *Linear Combination of Compatible Matrices:*

   The linear combination $cA + dB$ is the $m \times n$ matrix

   $$cA + dB = \begin{bmatrix} c\boldsymbol{a_1} + d\boldsymbol{b_1} & c\boldsymbol{a_2} + d\boldsymbol{b_2} & \cdots & c\boldsymbol{a_n} + d\boldsymbol{b_n} \end{bmatrix}.$$

   Equivalently, $cA + dB$ is the matrix whose (i, j)-entry is $(cA + dB)_{ij} = ca_{ij} + db_{ij}$.

Problem 2. Let $A = \begin{bmatrix} 2 & 3 & 6 \\ -2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -6 & 2 \\ 4 & 5 & 0 \end{bmatrix}$.

Explicitly find the following matrices: $A + B$, $5A$, $5A - 2B$.

2.1.4 Matrix Transpose

The *transpose* of the $m \times n$ matrix A, denoted by A^T ("A-transpose"), is the $n \times m$ matrix whose (i, j)-entry is a_{ji}: $\left(A^T\right)_{ij} = a_{ji}$. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

For convenience, let $\boldsymbol{\alpha_i} \in \mathbf{R}^n$ (for $i = 1, 2, \ldots, m$) be the columns of A^T. Then A is the matrix whose rows are the $\boldsymbol{\alpha_i}^T$'s. That is, if

$$A^T = \begin{bmatrix} \boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} & \cdots & \boldsymbol{\alpha_m} \end{bmatrix}, \text{ then } A = \begin{bmatrix} \boldsymbol{\alpha_1}^T \\ \vdots \\ \boldsymbol{\alpha_m}^T \end{bmatrix}.$$

<u>*Note*</u> that if A is a symmetric square matrix, then $A^T = A$.

2.1.5 Dot Product and Matrix Product

1. *Dot Product of Vectors in* $\mathbf{R}^n$*:*

   If $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in $\mathbf{R}^n$, then their dot product is the scalar

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{v}^T \boldsymbol{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

2. *Matrix Product of Compatible Matrices:*

   Let A be an $m \times n$ matrix and let $B = \left[\ \boldsymbol{b_1} \quad \boldsymbol{b_2} \quad \cdots \quad \boldsymbol{b_p} \ \right]$ be an $n \times p$ matrix. Then the matrix product, AB, is the $m \times p$ matrix whose j^{th} column is the matrix-vector product of A with the j^{th} column of B:

$$AB = \left[\ A\boldsymbol{b_1} \quad A\boldsymbol{b_2} \quad \cdots \quad A\boldsymbol{b_p} \ \right].$$

   Equivalently,

   AB is the $m \times p$ matrix whose (i, j)-entry is the dot product of the i^{th} row of A with the j^{th} column of B:

$$(AB)_{ij} = \boldsymbol{\alpha_i} \cdot \boldsymbol{b_j} = \boldsymbol{\alpha_i^T b_j} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj},$$

   where $\boldsymbol{\alpha_i^T}$ is the i^{th} row of A, for $i = 1, \ldots, m$.

Problem 3. In each case, explicitly find AB and BA, if possible. If both matrix products exist, determine if $AB = BA$.

(a) $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 4 \\ 4 & -7 & 0 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$.

2.1.6 Geometric Interpretation of Matrix Product

The matrix product can be interpreted as the standard matrix of the composition of two linear transformations, and thus has a geometric interpretation as well.

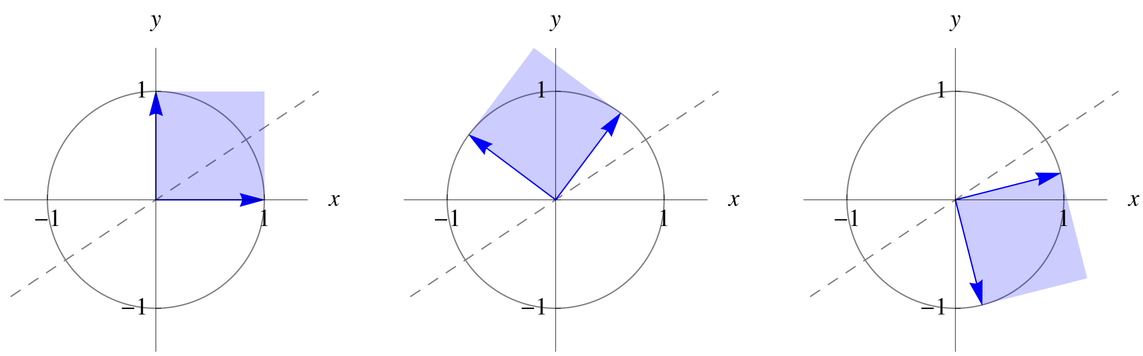
Specifically, if $T_B : \mathbf{R}^p \to \mathbf{R}^n$ has matrix B, $T_A : \mathbf{R}^n \to \mathbf{R}^m$ has matrix A, and $T = T_A \circ T_B$ is the composite function, then

$$T : \mathbf{R}^p \to \mathbf{R}^m \quad \text{defined by} \;\; T(\boldsymbol{x}) = T_A(T_B(\boldsymbol{x})) \;\; \text{has standard matrix } AB.$$

For example, let T_B correspond to rotation in 2-space about the origin through the positive angle $\theta = \arctan(4/3)$, and let T_A correspond to reflection in the line $y = 2x/3$. These transformations have the following rules:

$$T_B(\boldsymbol{x}) = B\boldsymbol{x} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \boldsymbol{x} \;\; \text{and} \;\; T_A(\boldsymbol{x}) = A\boldsymbol{x} = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix} \boldsymbol{x}.$$

The plots below illustrate the effect of the composition $T = T_A \circ T_B$ on the unit square in the plane: $\mathcal{R}$ is first rotated through angle θ and then reflected in the line.



$\mathcal{R} = [0,1] \times [0,1]$ $\qquad$ $T_B(\mathcal{R})$ $\qquad$ $T_A(T_B(\mathcal{R}))$

Problem 4. Find the standard matrix of the linear transformation $T = T_A \circ T_B$.

2.1.7 Properties of the Basic Operations

Some properties of the basic operations are given below. In each case, the sizes of the matrices are assumed to be compatible.

1. *Commutative:* $A + B = B + A$.

2. *Associative:* $A + (B + C) = (A + B) + C$ and $A(BC) = (AB)C$.

3. *Distributive:* $A(B + C) = (AB) + (AC)$, $(B + C)A = (BA) + (CA)$.

4. *Scalar Multiples:* Given $c \in \mathbf{R}$:

$$c(AB) = (cA)B = A(cB),\ c(A + B) = (cA) + (cB) \text{ and } (cA)^T = cA^T.$$

5. *Identity:* $I_m A = A = A I_n$ when A is an $m \times n$ matrix.

6. *Transpose of Transpose:* $\left(A^T\right)^T = A$.

7. *Transpose of Sum:* $(A + B)^T = A^T + B^T$.

8. *Transpose of Product:* $(AB)^T = B^T A^T$.

Note on Commutativity:

Matrix sum is commutative but matrix product is *not*.

In fact, as we have already seen, it is possible that AB is defined but that BA is undefined.

Note on Associativity:

The associative law for products says that products like ABC are well-defined.

It is up to you whether you would like to compute AB first and then multiply by C on the right, or to compute BC first and then multiply by A on the left.

2.1.8 Powers of Square Matrices and Finding Patterns

Let A be a square matrix of order n. We often use the following shorthand notation to represent the products of A with itself:

$$A^2 = AA,\ A^3 = AAA,\ A^4 = AAAA, \text{ and so forth.}$$

These products are called the *powers* of the square matrix A.

Problem 5. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. Find A^2, A^3, A^4 and A^{100}.

Problem 6. Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. (a) Find A^2, A^3, A^4 and A^p where p is a positive integer.

(b) What properties are suggested by your solution to part (a)? Be as complete as possible.

Problem 7. Let $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$. (a) Find AB and BA.

(b) What properties are suggested by your solution to part (a)? Be as complete as possible.

2.1.9 Inverses of Square Matrices

Let A be a square matrix of order n. Then

1. *Invertible/Not Invertible:*

   A is said to be invertible if there exists a square matrix C satisfying

   $$AC = CA = I_n \quad \text{where } I_n \text{ is the identity matrix of order } n.$$

   Otherwise, A is said to be not invertible.

2. *Singular/NonSingular:*

   Square matrices that are not invertible are also called singular matrices.

   Invertible square matrices are also called nonsingular matrices.

Problem 8. Use the properties of the basic matrix operations (given in Section 2.1.7, page 9) to demonstrate that inverses are unique.

Notation for Inverses:

The notation A^{-1} (read "A inverse") is used to denote the *matrix inverse* of A.

Finding the inverse of a square matrix (or determining that an inverse does not exist) can often involve many computations.

Here are a few special cases:

1. *Powers:*

   If A is a square matrix and p is the smallest positive integer satisfying $A^p = I$, then A is invertible with inverse $A^{-1} = A^{p-1}$. In symbols,

   $$I = A^p = A\left(A^{p-1}\right) = \left(A^{p-1}\right)A \implies A^{-1} = A^{p-1}.$$

2. *2-by-2 Matrices:*

   Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix. Then

   $$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ when } ad - bc \neq 0.$$

   If $ad - bc = 0$, then A does not have an inverse.

   (Note: The quantity $ad - bc$ is known as the *determinant* of A.)

3. *Diagonal Matrices:*

   Let A be a diagonal matrix with nonzero diagonal elements (i.e, with $a_{ii} \neq 0$ for all i). Then A is invertible with inverse

$$A^{-1} = \begin{bmatrix} 1/a_{11} & 0 & \cdots & 0 \\ 0 & 1/a_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}.$$

   That is, A^{-1} is the diagonal matrix whose diagonal elements are the reciprocals of the diagonal elements of A. If $a_{ii} = 0$ for some i, then A is not invertible.

For example,

1. If $A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, then from the work we did on page 10, we know that $A^3 = I$.

   Thus, the inverse of A is $A^{-1} = A^2 = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$.

2. If $A = \begin{bmatrix} -7 & 8 \\ 2 & -2 \end{bmatrix}$, then the determinant of A is ____________________

   and the inverse of A is $A^{-1} = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$.

3. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$.

4. An example of a square matrix $A \neq I$ satisfying $A^{-1} = A$ is (please complete)

2.1.10 Properties of Inverses

Let A and B be invertible square matrices of order n and let c be a nonzero constant. Then

1. *Inverse of Identity:* $I_n^{-1} = I_n$ for each n.
2. *Inverse of Inverse:* $\left(A^{-1}\right)^{-1} = A$.
3. *Inverse of Product:* $(AB)^{-1} = B^{-1}A^{-1}$.
4. *Inverse of Scalar Multiple:* $(cA)^{-1} = (1/c)A^{-1}$.
5. *Inverse of Transpose:* $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Demonstration of Property 3. It is instructive to demonstrate the third property. Assume that A and B are invertible square matrices of size n and let $C = B^{-1}A^{-1}$. We need to show:

$$(1)\ (AB)C = I_n, \text{ and } (2)\ C(AB) = I_n.$$

Now (complete the proof),

2.1.11 Inverses and Solving Systems of Equations

If the coefficient matrix of an n-by-n system of linear equations is invertible, then the unique solution to the system can be found using inverses. Specifically,

> ***Theorem (Uniqueness Theorem).*** Let A be an invertible $n \times n$ matrix. Then the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ for any $\boldsymbol{b} \in \mathbf{R}^n$.

Proof of Uniqueness Theorem:

The uniqueness theorem is actually quite simple to prove since

$$A\boldsymbol{x} = \boldsymbol{b} \Rightarrow A^{-1}(A\boldsymbol{x}) = A^{-1}\boldsymbol{b} \Rightarrow (A^{-1}A)\boldsymbol{x} = A^{-1}\boldsymbol{b} \Rightarrow \boldsymbol{x} = A^{-1}\boldsymbol{b},$$

where the last implication follows since $A^{-1}A = I$.

Problem 9. Use inverses to solve the 2-by-2 system: $-7x_1 + 8x_2 = -3$ and $2x_1 - 2x_2 = 5$.

Finding the inverse and solving n linear systems. An algorithm for finding the inverse can be developed using the following observations.

Let A be an invertible matrix of order n and $I_n = \left[\begin{array}{cccc} \boldsymbol{e_1} & \boldsymbol{e_2} & \cdots & \boldsymbol{e_n} \end{array}\right]$ be the identity matrix of order n. For convenience, let $C = A^{-1}$ in the discussion below.

Then $AC = I_n$ can be written as follows:

$$AC = \begin{bmatrix} A\boldsymbol{c_1} & A\boldsymbol{c_2} & \cdots & A\boldsymbol{c_n} \end{bmatrix} = \begin{bmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \cdots & \boldsymbol{e_n} \end{bmatrix}.$$

Thus, the j^{th} column of the inverse is the solution to the system

$$A\boldsymbol{c_j} = \boldsymbol{e_j}, \quad \text{for} \quad j = 1, 2, \ldots, n.$$

To turn the observations (which assume that the inverse exists) into an algorithm for finding the inverse, we need the following facts:

Fact 1: The $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n.

Further, if A is row equivalent to I_n, then any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1}.

Fact 2: Suppose that A and D are square matrices of order n. Then,

if $AD = I_n$, then A is an invertible matrix.

Algorithm for finding an inverse. Let A be an $n \times n$ matrix. Form the $n \times 2n$ augmented matrix whose last n columns are the columns of the identity matrix. If A is row equivalent to the identity matrix, then the $n \times 2n$ augmented matrix will be transformed as follows:

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] \sim \cdots \sim \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$

That is, the last n columns will be the inverse of A. Otherwise, A does not have an inverse.

For example, let $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix}$. Then $\left[\begin{array}{c|c} A & I_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} \\ \frac{3}{2} & 1 & 0 \end{bmatrix}.$$

Problem 10. Find the inverse of $A = \begin{bmatrix} 6 & 1 & 2 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ or state that the inverse does not exist.

Problem 11. Find the inverse of $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ or state that the inverse does not exist.

2.1.12 Inverses and Elementary Matrices

An *elementary matrix*, E, is one obtained by performing a single elementary row operation on an identity matrix. Since each elementary row operation is reversible, each elementary matrix is invertible.

For example, beginning with the identity matrix of order 2, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

1. If the operation is $R_2 \leftarrow R_2 + 2R_1$ (with reverse operation $R_2 \leftarrow R_2 - 2R_1$), then

$$E = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \text{ and } E^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

2. If the operation is $R_2 \leftarrow 3R_2$ (with reverse operation $R_2 \leftarrow \frac{1}{3}R_2$), then

$$E = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix} \text{ and } E^{-1} = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}.$$

3. If the operation is $R_2 \leftrightarrow R_1$ (with reverse operation $R_2 \leftrightarrow R_1$), then

$$E = E^{-1} = \begin{bmatrix} \_\_ & \_\_ \\ \_\_ & \_\_ \end{bmatrix}.$$

Left multiplication performs operation. An elementary row operation performed on a matrix A corresponds to *left* multiplication by the corresponding elementary matrix.

For example, if $A = \begin{bmatrix} 3 & 1 & 0 \\ -6 & 0 & -4 \end{bmatrix}$ and E is the first matrix above, then the product

$$EA = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ -6 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -4 \end{bmatrix}$$

has the effect of replacing R_2 by $R_2 + 2R_1$.

Relationship to inverse of A. Let A be an $n \times n$ matrix. Suppose that A can be transformed to the identity matrix using a sequence of p elementary row operations with elementary matrices E_1, E_2, ..., E_p. Then

$$(E_p(E_{p-1} \cdots (E_2(E_1A)))) = (E_pE_{p-1} \cdots E_2E_1)A = I_n \quad \Rightarrow \quad A^{-1} = E_pE_{p-1} \cdots E_2E_1.$$

Thus, A^{-1} is the product of the elementary matrices used to transform A to I_n.

Problem 12. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix}$. The work on page 16 implies that A can be transformed to the identity matrix using the following four elementary row operations:

$$(1)\ R_2 \leftarrow R_2 + \frac{3}{2}R_1, \quad (2)\ R_2 \leftrightarrow R_3, \quad (3)\ R_2 \leftarrow \frac{1}{4}R_2, \quad (4)\ R_1 \leftarrow \frac{1}{2}R_1$$

Find the corresponding elementary matrices E_1, E_2, E_3 and E_4, and verify that the inverse of A equals the product $E_4E_3E_2E_1$.

2.2 Factoring and Blocking

2.2.1 Factorization; *LU*-Factorization

1. *Factorization of Matrix:*

   A factorization of the $m \times n$ matrix A is an equation that expresses A as the product of two or more matrices.

2. *LU-Factorization of Matrix:*

   The $m \times n$ matrix A has an LU-factorization if it is row equivalent to an echelon form matrix using row replacements only. In this case, $A = LU$ where

   (a) L is an $m \times m$ lower triangular matrix with 1's on the diagonal and

   (b) U is an $m \times n$ echelon form matrix.

Notes on LU-Factorizations:

(1) Since L has 1's on the diagonal, it is often called a *unit lower triangular matrix.*

(2) If A is square, then the echelon form matrix U is upper triangular matrix.

Problem 1. Use the fact that $A = \begin{bmatrix} 3 & 1 & 0 \\ -6 & 0 & -4 \\ 0 & 6 & -10 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -4 \\ 0 & 6 & -10 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & 2 \end{bmatrix}$ using operations $R_2 \to R_2 + 2R_1$ followed by $R_3 \to R_3 - 3R_2$ to find an LU-factorization of A.

Problem 2. Use the fact that $A = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 12 & 16 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ using operations (1) $R_2 \to R_2 - 2R_1$, (2) $R_3 \to R_3 - 3R_1$ and (3) $R_3 \to R_3 - 4R_2$ to find an LU-factorization of A.

Problem 3. What patterns, if any, do you see in the solutions to the problems above?

Solving linear systems efficiently. Suppose that $A = LU$ where L is a unit lower triangular matrix and U is an echelon form matrix. Then the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ can be solved efficiently as follows. Consider

$$\boldsymbol{b} = A\boldsymbol{x} = (LU)\boldsymbol{x} = L(U\boldsymbol{x}) = L\boldsymbol{y}, \quad \text{where} \quad \boldsymbol{y} = U\boldsymbol{x}.$$

The efficient method is

1. Solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$.
2. Solve $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$.

For example, let $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and assume that

$$A = LU = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} \text{ is the } LU\text{-factorization of } A.$$

To solve $A\boldsymbol{x} = \boldsymbol{b}$ using the method above,

$$(1) \quad [\, L \mid \boldsymbol{b} \,] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ -2 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \end{array}\right] \ \Rightarrow \ \boldsymbol{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$(2) \quad [\, U \mid \boldsymbol{y} \,] = \left[\begin{array}{cc|c} 2 & -3 & 2 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 0 & 8 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right] \ \Rightarrow \ \boldsymbol{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

is the solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Problem 1, continued. Let $A = \begin{bmatrix} 3 & 1 & 0 \\ -6 & 0 & -4 \\ 0 & 6 & -10 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$. Use the LU-factorization of A we computed earlier to (1) solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$ and (2) solve $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$.

Efficiency of LU-factorization. The method above solves the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ by (1) factoring A, (2) using the equivalent of forward substitution to find $\boldsymbol{y}$ and (3) using the equivalent of backward substitution to find $\boldsymbol{x}$.

For n-by-n systems, for example,

1. solving $A\boldsymbol{x} = \boldsymbol{b}$ directly takes roughly n^3 operations, while
2. solving $L\boldsymbol{y} = \boldsymbol{b}$ and $U\boldsymbol{x} = \boldsymbol{y}$ together take roughly n^2 operations.

So the method is useful when n is very large and the same coefficient matrix will be used with many different $\boldsymbol{b}$'s. Large-scale heat flow and large-scale air flow problems are examples of situations where the method above is useful.

In addition, since scalings are held off as long as possible, the method is useful in situations where scalings could cause numerical roundoff problems.

2.2.2 Permutation Matrices; PA=LU Form

A *permutation matrix* P is an elementary matrix corresponding to a row interchange, or a product of elementary matrices corresponding to multiple row interchanges.

For example, if $n = 4$,

- The permutation matrix corresponding to $R_1 \leftrightarrow R_3$ is

- The permutation matrix corresponding to (1) $R_1 \leftrightarrow R_3$ followed by (2) $R_3 \leftrightarrow R_4$ is

Use of permutation matrices in factoring. If row interchanges are needed to transform A to echelon form, then an LU-factorization is not possible. But, we can write

$$PA = LU, \text{ where } P \text{ is a permutation matrix.}$$

For example, let $A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -8 & 1 \end{bmatrix}$. Since

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -8 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -8 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -8 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

using operations (1) $R_1 \leftrightarrow R_3$, (2) $R_2 \leftarrow R_2 - 3R_1$, and (3) $R_4 \leftarrow R_4 + 4R_3$,

we can construct the $PA = LU$ form as follows:

2.2.3 Partitioned Matrix, Block Matrix; Inverses in Special Cases

A *partitioned matrix* (or *block matrix*) is one whose entries have been grouped into smaller matrices using vertical and horizontal cuts through the matrix.

For example,

$$A = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \\ \hline 0 & 0 & 6 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } B = \left[\begin{array}{rr|rr} 3 & 1 & 1 & 0 \\ -4 & 3 & 0 & 1 \\ \hline 2 & -4 & 0 & 0 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

are examples of 2×2 block matrices.

Note that

1. Matrices of the same size (that is, with the same numbers of rows and columns) and blocked in the same way (that is, so that corresponding blocks have the same numbers of rows and columns) can be added by adding blocks.

2. Matrices compatible for multiplication and blocked in compatible ways can be multiplied in blocks. For example, for compatible 2×2 block matrices:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

   as long as all multiplications and additions are valid.

For example, the 2×2 block matrices above are compatible for multiplication:

$$AB = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & I \\ B_{21} & O \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11} \\ A_{22}B_{21} & O \end{bmatrix} = \left[\begin{array}{cc|cc} \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \\ \hline \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{array}\right]$$

where O is a zero matrix of an appropriate size and I is an identity matrix of an appropriate size; using block structure simplifies the process.

Problem 4. Let $A = \left[\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ \hline 1 & 2 & 3 & 6 \\ 3 & 4 & 9 & 12 \end{array}\right]$ and $B = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ \hline -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{array}\right]$.

Compute the product AB. Take advantage of the block structure as much as possible.

2×2 block diagonal matrices, and their inverses. Let A be a square matrix of order n. Then A is a 2×2 *block diagonal matrix* if

$$A = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}, \text{ where } A_{11} \text{ and } A_{22} \text{ are square matrices of orders } p \text{ and } q,$$

respectively, and each O is a zero matrix of an appropriate size.

For example, the following matrix is a 2×2 block diagonal matrix

$$A = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} = \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 & 0 \\ 7 & 8 & 9 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 & 4 \end{array}\right]$$

where A_{11} is a square matrix of order 3 and A_{22} is a square matrix of order 2.

Inverses of block diagonal matrices are block diagonal. Specifically,

Theorem (Inverses). Let A be a 2×2 block diagonal matrix. Then

A is invertible if and only if both A_{11} and A_{22} are invertible.

Further, if A is invertible, then $A^{-1} = \begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix}$.

It is instructive to write out the proof of the theorem on inverses:

($\Rightarrow$): Suppose that A is invertible and let $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ be the inverse of A. Then $AC = I_n$ implies

$$\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11}C_{11} & A_{11}C_{12} \\ A_{22}C_{21} & A_{22}C_{22} \end{bmatrix} = \begin{bmatrix} I_p & O \\ O & I_q \end{bmatrix}.$$

Now, (complete the proof)

($\Leftarrow$): Suppose that A_{11} and A_{22} are invertible and let $C = \begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix}$. Then a direct calculation shows that

$$AC = CA = \begin{bmatrix} I_p & O \\ O & I_q \end{bmatrix} = I_n. \text{ Thus, } C = A^{-1}.$$

Problem 5. In each case, use block methods to find the inverse of A:

$$(1)\ A = \begin{bmatrix} 5 & 3 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \qquad (2)\ A = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 6 & 4 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 2 & 3 \end{bmatrix}.$$

2×2 block upper triangular matrices, and their inverses. Let A be a square matrix of order n. Then A is a 2×2 *block upper triangular matrix* if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}, \text{ where } A_{11} \text{ and } A_{22} \text{ are square matrices of orders } p \text{ and } q,$$

respectively, and O is a zero matrix of an appropriate size.

The following theorem gives us the form of the inverse of a block upper triangular matrix, when it exists. Specifically,

> ***Theorem (Inverses).*** Let A be a 2×2 block upper triangular matrix as above. Then
>
> $$A \text{ is invertible if and only if both } A_{11} \text{ and } A_{22} \text{ are invertible.}$$
>
> Further, if A is invertible, then $A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ O & A_{22}^{-1} \end{bmatrix}$.

Problem 6. (a) In each case, use block methods to find the inverse of A:

(1) $A = \begin{bmatrix} 5 & 3 & 1 \\ 6 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(2) $A = \begin{bmatrix} 5 & 3 & 0 & 1 \\ 6 & 4 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 2 & 3 \end{bmatrix}$.

(3) $A = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -4 & 2 & 7 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(4) $A = \begin{bmatrix} -1 & 0 & 2 & 1 & 3 \\ 0 & -1 & -4 & 2 & 7 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

(b) Suggest special cases of the inverse formula for 2×2 block upper triangular matrices based on the results of the last two inverses in part (a). Be as complete as possible.

2.3 Determinants

2.3.1 Determinants and Areas

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix, then the *determinant* of A is the number

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Let $A = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} \end{bmatrix}$. Analytic geometry can be used to show that the area of the parallelogram with corners $\boldsymbol{O}$, $\boldsymbol{a_1}$, $\boldsymbol{a_2}$ and $\boldsymbol{a_1} + \boldsymbol{a_2}$ is the absolute value of the determinant:

$$\text{Area of Parallelogram} = |det(A)| = |ad - bc|.$$

For example, if

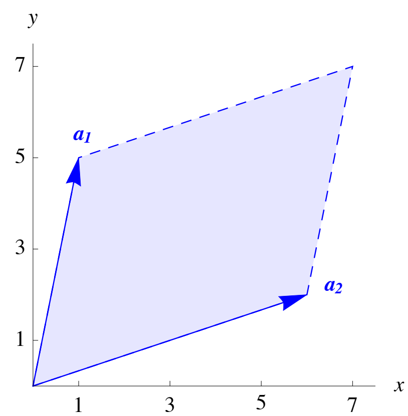
$$A = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix},$$

then

(1) $det(A) =$ ________________________, and

(2) the area of the parallelogram shown in

the plot is ____________________________



2.3.2 Recursive Definition of Determinants; Volumes

If A is an $n \times n$ matrix, then the determinant of A can be defined recursively as an alternating sum of multiples of determinants of smaller matrices. Using expansion in the first row of A:

$$det(A) = |A| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \; det(A_{1j}),$$

where A_{1j} is the submatrix obtained by eliminating the 1^{st} row and j^{th} column of A.

The form for a 3×3 matrix is:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

The computation is completed once you fill in the determinants of each 2×2 submatrix, and simplify the results.

For example, if $A = \begin{bmatrix} 2 & 12 & 1 \\ 10 & 4 & 1 \\ 2 & 2 & 8 \end{bmatrix}$, then the determinant of A is

Determinants and volumes. Let $A = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \end{bmatrix}$ be a 3×3 matrix. Analytic geometry can be used to show that the volume of the solid with corners $\boldsymbol{O}$, $\boldsymbol{a_1}$, $\boldsymbol{a_2}$, $\boldsymbol{a_3}$, $\boldsymbol{a_1} + \boldsymbol{a_2}$, $\boldsymbol{a_1} + \boldsymbol{a_3}$, $\boldsymbol{a_2} + \boldsymbol{a_3}$ and $\boldsymbol{a_1} + \boldsymbol{a_2} + \boldsymbol{a_3}$ is the absolute value of the determinant:

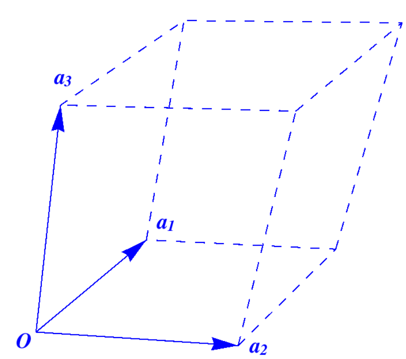
$$\text{Volume of Parallelopiped} = |det(A)|.$$

For example, if

$$A = \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \end{bmatrix} = \begin{bmatrix} 2 & 12 & 1 \\ 10 & 4 & 1 \\ 2 & 2 & 8 \end{bmatrix},$$

then the volume of the parallelopiped shown

in the plot is ___________________________



Theorem (Equivalent Recursive Definitions). The determinant of the square matrix A can be obtained recursively by expanding in any row or in any column. That is,

$$det(A) = |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\ det(A_{ij}) \text{ for any fixed row } i \text{ and}$$

$$det(A) = |A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij}\ det(A_{ij}) \text{ for any fixed column } j,$$

where A_{ij} is the submatrix obtained by removing the i^{th} row and j^{th} column of A.

Problem 1. Find each determinant using expansion.

$(a)\ \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} =$ ____

$(b)\ \begin{vmatrix} 1 & 0 & 0 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} =$ ____

$(c)\ \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} =$ ____

$(d)\ \begin{vmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{vmatrix} =$ ____

$(e)\ \begin{vmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & e & f & 0 \\ g & h & i & j \end{vmatrix} =$ ____

2.3.3 Properties of Determinants

Let A, B and E be square matrices of order n. Then

1. $det\left(A^T\right) = det(A)$.
2. $det(AB) = det(A)det(B)$.
3. If A is a triangular matrix, then $det(A) = a_{11}a_{22}\cdots a_{nn}$.
4. If E is an elementary matrix for a row replacement, then $det(E) = 1$.
5. If E is an elementary matrix for a row interchange, then $det(E) = -1$.
6. If E is an elementary matrix for scaling a row by c, then $det(E) = c$.

Problem 2. Let A be an invertible square matrix. Use the properties of determinants to explain why the determinant of A must be nonzero.

Problem 3. Let P be a permutation matrix (that is, a product of elementary matrices for row interchanges). Demonstrate that the determinant of P can only take two values.

Problem 4. Let A be a square matrix of order n. In each case, find a simple formula for the determinant of A:

(1) A has an LU-factorization.

(2) A does not have an LU-factorization, but a factorization of the form $PA = LU$ is possible.

2.3.4 Determinants and Elementary Row Operations

The following theorem relates elementary row operations to determinants.

> ***Theorem (Row Operations and Determinants).*** Let A be a square matrix. Then
>
> 1. If B is obtained from A by a single row replacement, then $det(B) = det(A)$.
> 2. If B is obtained from A by a single row interchange, then $det(B) = -det(A)$.
> 3. If B is obtained from A by multiplying a row by c, then $det(B) = c\ det(A)$.

To see why this theorem is true, assume that $B = EA$, where E is the elementary matrix for the given operation. Then (please complete)

This theorem (and the properties of determinants) can be used to develop an alternative method for finding a determinant based on row operations.

For example,

$$\begin{vmatrix} 1 & -4 & 2 \\ -1 & 7 & 0 \\ -2 & 8 & -9 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ -2 & 8 & -9 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = \underline{\hspace{6cm}}$$

(In step 1, the row replacement $R_2 \leftarrow R_2 + R_1$ does not change the value of the determinant. Likewise, in step 2, the row replacement $R_3 \leftarrow R_3 + 2R_1$ does not change the value of the determinant. Since the third matrix is in echelon form, its determinant is easy to compute.)

Problem 5. Find each determinant using the method based on row operations.

$$(a) \begin{vmatrix} 2 & -8 & 4 \\ -1 & 7 & 0 \\ -2 & 8 & -9 \end{vmatrix} = \underline{\hspace{12cm}}$$

$$(b) \begin{vmatrix} 3 & 6 & 6 \\ 1 & 2 & 3 \\ 1 & 4 & 4 \end{vmatrix} = \underline{\hspace{12cm}}$$

$$(c) \begin{vmatrix} 0 & 0 & -2 & 2 \\ 1 & 0 & 1 & 2 \\ -1 & 1 & 2 & 1 \\ 0 & 2 & -3 & 0 \end{vmatrix} = \underline{\hspace{12cm}}$$

Methods for finding determinants include expansion in rows/columns; using elementary row operations to reduce the matrix to echelon form; or some combination of these approaches. Properties of determinants can also be used in conjunction with these methods.

Problem 7. Let

$$M = \begin{bmatrix} A & O \\ C & D \end{bmatrix} = \begin{bmatrix} A & O \\ O & I_q \end{bmatrix} \begin{bmatrix} I_p & O \\ C & D \end{bmatrix}$$

be a 2×2 block triangular matrix. Use this factorization and the properties of determinants to demonstrate that the determinant of M is $det(M) = det(A)det(D)$.

2.3.5 Determinants and Invertibility

Theorem (Invertible Matrices and Determinants). Let A be a square matrix. Then

A is invertible if and only if the determinant of A is nonzero.

It is instructive to write out the proof of the theorem relating invertibility to determinants:

(⇒): This first part of the theorem was demonstrated in Problem 2 (on page 33), using the properties of determinants listed on that page.

(⇐): Assume that $det(A) \neq 0$ and that A can be transformed to echelon form using k elementary operations. That is, assume the

$$E_k E_{k-1} \cdots E_1 A = U,$$

where each E_i is an elementary matrix and U is an echelon form matrix.

Now (complete this part),

To illustrate the theorem relating determinants and invertibility, let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{bmatrix}, \quad \text{where } c \text{ and } d \text{ are constants.}$$

Since $det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & c & c^2 \\ 1 & d & d^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & c-1 & c^2-1 \\ 0 & d-1 & d^2-1 \end{vmatrix} = (c-1)(d-1) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & c+1 \\ 0 & 1 & d+1 \end{vmatrix} =$

$$= (c-1)(d-1) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & c+1 \\ 0 & 0 & d-c \end{vmatrix} = \underline{\hspace{6cm}},$$

we know that A is invertible iff ____.

<u>*Note: Vandermonde Matrices*</u>
The matrix in the illustration above is an example of a Vandermonde matrix.