MATH2210 Notebook 3 **Spring 2018**

prepared by Professor Jenny Baglivo

3 MATH2210 Notebook 3

This notebook is concerned with vector space concepts in linear algebra. The notes correspond to material in Chapter 4 of the Lay textbook.

3.1 Vector Spaces and Subspaces

Recall that if $\{\boldsymbol{v}, \boldsymbol{w}\} \subseteq \mathbf{R}^n$ is a linearly independent set, then the span of these vectors,

$$\text{Span}\{\boldsymbol{v}, \boldsymbol{w}\} = \{c\boldsymbol{v} + d\boldsymbol{w} \mid c, d \in \mathbf{R}\} \subseteq \mathbf{R}^n,$$

corresponds to a plane through the origin in $\mathbf{R}^n$.

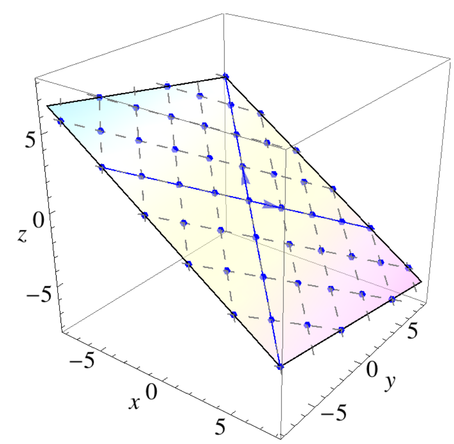
The span of a linearly independent set $\{\boldsymbol{v}, \boldsymbol{w}\}$ of 2 vectors is, in many ways, "just like" $\mathbf{R}^2$ since points in the span are uniquely determined using the two coordinates (c, d).

For example, since

$$\{\boldsymbol{v}, \boldsymbol{w}\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\} \subseteq \mathbf{R}^3$$

is a linearly independent set in 3-space, we know that its span is a plane through the origin in 3-space, as illustrated to the right.

The grid shown on the plane corresponds to points where either c or d is an integer.



This chapter explores objects that have algebraic properties "just like" $\mathbf{R}^k$ for some k, and generalizes these objects.

Question: Do you remember how to show that $\{\boldsymbol{v}, \boldsymbol{w}\}$ is a linearly independent set?

3.1.1 Vector Spaces Over the Reals

A vector space V is a nonempty set of objects (called *vectors*) on which are defined the operations of *addition* and *scalar multiplication* satisfying the following rules (axioms):

1. *Closed Under Vector Addition:* For each $\boldsymbol{x}, \boldsymbol{y} \in V$, $\boldsymbol{x} + \boldsymbol{y} \in V$.
2. *Closed Under Scalar Multiplication:* For each $\boldsymbol{x} \in V$ and $c \in \mathbf{R}$, $c\boldsymbol{x} \in V$.
3. *Properties of Addition:* Vector addition is commutative and associative.
4. *Zero Vector:* There is a zero vector, $\boldsymbol{O} \in V$, that satisfies the following property:
$$\boldsymbol{x} + \boldsymbol{O} = \boldsymbol{O} + \boldsymbol{x} = \boldsymbol{x} \text{ for all } \boldsymbol{x} \in V.$$
5. *Additive Inverse:* For each $\boldsymbol{x} \in V$ there is a $\boldsymbol{y} \in V$ satisfying $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x} = \boldsymbol{O}$.
6. *Properties of Scalar Multiplication:* Scalar multiplication satisfies
$$c(\boldsymbol{x} + \boldsymbol{y}) = (c\boldsymbol{x}) + (c\boldsymbol{y}), \quad (c+d)\boldsymbol{x} = (c\boldsymbol{x}) + (d\boldsymbol{x}), \quad (cd)\boldsymbol{x} = c(d\boldsymbol{x}), \quad 1\boldsymbol{x} = \boldsymbol{x},$$
for all $c, d \in \mathbf{R}$, $\boldsymbol{x}, \boldsymbol{y} \in V$.

***Example:* $\mathbf{R}^n$.** For each n, $\mathbf{R}^n$ with the usual definitions of vector sum and scalar product is a vector space. In fact, $\mathbf{R}^n$ is our "model" vector space.

***Example:* $\boldsymbol{M_{m\times n}}$.** Let $\boldsymbol{M_{m\times n}}$ be the collection of all $m\times n$ matrices with the usual definitions of matrix sum and scalar product. Then $\boldsymbol{M_{m\times n}}$ satisfies the vector space axioms, where the "zero vector" of the matrix space is the zero matrix O of size $m \times n$.

A particular example we will use often in illustrations is the vector space of 2×2 matrices:

$$\boldsymbol{M_{2\times 2}} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbf{R} \right\}$$

with the usual definitions of matrix sum and scalar product.

In many ways, the vector space $\boldsymbol{M_{2\times 2}}$ behaves like the vector space $\mathbf{R}^4$ with respect to the operations of addition and scalar multiplication, where

$$\mathbf{R}^4 = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbf{R} \right\}$$

with the usual definitions of vector sum and scalar product.

More generally, the vector spaces $\boldsymbol{M_{m\times n}}$ and $\mathbf{R}^{mn}$ have a lot in common.

Example: P_n. Let $\boldsymbol{P_n}$ be the collection of polynomials of degree at most n. A polynomial $\boldsymbol{p} \in \boldsymbol{P_n}$ when it takes the following form:

$$\boldsymbol{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \text{ for all } t \in \mathbf{R},$$

for fixed constants $a_0, a_1, \ldots, a_n \in \mathbf{R}$. Operations on $\boldsymbol{P_n}$ are defined as follows:

1. *Addition:* Given $\boldsymbol{p_1}, \boldsymbol{p_2} \in \boldsymbol{P_n}$, then $\boldsymbol{p_1} + \boldsymbol{p_2}$ is the polynomial whose value at each t is the sum of the values $\boldsymbol{p_1}(t)$ and $\boldsymbol{p_2}(t)$:

   $$(\boldsymbol{p_1} + \boldsymbol{p_2})(t) = \boldsymbol{p_1}(t) + \boldsymbol{p_2}(t) \text{ for each } t \in \mathbf{R}.$$

   (You add polynomials by adding their values at each t.)

2. *Scalar Multiplication:* Given$\boldsymbol{p} \in \boldsymbol{P_n}$ and $k \in \mathbf{R}$, then $k\boldsymbol{p}$ is the polynomial whose value at each t is k times the value $\boldsymbol{p}(t)$:

   $$(k\boldsymbol{p})(t) = k\,(\boldsymbol{p}(t)) \text{ for each } t \in \mathbf{R}.$$

   (You scale a polynomial by scaling its value at each t.)

The zero vector of $\boldsymbol{P_n}$ is the constant function all of whose values are zero:

$$\boldsymbol{O}(t) = 0 \text{ for all } t \in \mathbf{R}.$$

In many ways, the vector space $\boldsymbol{P_n}$ behaves like the vector space $\mathbf{R}^{n+1}$ with respect to the operations of addition and scalar multiplication, where

$$\mathbf{R}^{n+1} = \left\{ \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \;\middle|\; a_0, a_1, \ldots, a_n \in \mathbf{R} \right\}$$

with the usual definitions of vector sum and scalar product.

<u>Note</u> that $\boldsymbol{P_n}$ consists of polynomials whose degrees are <u>at most n</u>, not exactly n. Thus,

$$\boldsymbol{P_0} \subset \boldsymbol{P_1} \subset \cdots \subset \boldsymbol{P_{n-1}} \subset \boldsymbol{P_n}.$$

Thus, the vector space of constant functions,

$$\boldsymbol{P_0} = \{\boldsymbol{p}(t) = a \mid a \in \mathbf{R}\},$$

is contained within the vector space of constant and linear functions,

$$\boldsymbol{P_1} = \{\boldsymbol{p}(t) = a + bt \mid a, b \in \mathbf{R}\},$$

is contained within the vector space of constant, linear and quadratic functions,

$$\boldsymbol{P_2} = \{\boldsymbol{p}(t) = a + bt + ct^2 \mid a, b, c \in \mathbf{R}\},$$

and so forth.

***Example:* $\boldsymbol{C[a,b]}$.** Let $[a,b] \subset \mathbf{R}$ be an interval, and let $\boldsymbol{C[a,b]}$ be the collection of continuous real-valued functions with domain $[a,b]$. A convenient way to write $\boldsymbol{C[a,b]}$ is as follows:

$$\boldsymbol{C[a,b]} = \{\boldsymbol{f} : [a,b] \to \mathbf{R} \mid \boldsymbol{f} \text{ is continuous}\}.$$

Operations on $\boldsymbol{C[a,b]}$ are defined as follows:

1. *Addition:* Given $\boldsymbol{f_1}, \boldsymbol{f_2} \in \boldsymbol{C[a,b]}$, then $\boldsymbol{f_1} + \boldsymbol{f_2}$ is the function whose value at each $x \in [a,b]$ is the sum of the values $\boldsymbol{f_1}(x)$ and $\boldsymbol{f_2}(x)$:

   $$(\boldsymbol{f_1} + \boldsymbol{f_2})(x) = \boldsymbol{f_1}(x) + \boldsymbol{f_2}(x) \text{ for each } x \in [a,b].$$

   (You add functions by adding their values at each x.)

2. *Scalar Multiplication:* Given$\boldsymbol{f} \in \boldsymbol{C[a,b]}$ and $k \in \mathbf{R}$, then $k\boldsymbol{f}$ is the function whose value at each x is k times the value $\boldsymbol{f}(x)$:

   $$(k\boldsymbol{f})(x) = k\,(\boldsymbol{f}(x)) \text{ for each } x \in [a,b].$$

   (You scale functions by scaling their values at each x.)

The zero vector of $\boldsymbol{C[a,b]}$ is the constant function all of whose values are zero:

$$\boldsymbol{O}(x) = 0 \text{ for all } x \in [a,b].$$

The vector space of continuous functions on $[a,b]$ does not behave like the vector space $\mathbf{R}^n$ with respect to the operations of addition and scalar multiplication for any n.

Note that the vector space of continuous functions on $[a,b]$ contains many interesting subsets, many of which are themselves vector spaces.

Examples include the following:

1. *Polynomials with Restricted Domains:*

   Let $\boldsymbol{P_n[a,b]}$ be the collection of polynomial functions of degree at most n with domains restricted to the interval $[a,b]$. Since polynomial functions are continuous, we know that

   $$\boldsymbol{P_0[a,b]} \subset \boldsymbol{P_1[a,b]} \subset \cdots \subset \boldsymbol{P_n[a,b]} \subset \cdots \subset \boldsymbol{C[a,b]}.$$

   Each $\boldsymbol{P_n[a,b]}$ satisfies the vector space axioms.

2. *Differentiable Functions on $[a,b]$:*

   Let $\boldsymbol{D[a,b]}$ be the collection of differentiable real-valued functions with domain $[a,b]$. Since differentiable functions are continuous, we know that

   $$\boldsymbol{D[a,b]} \subset \boldsymbol{C[a,b]}.$$

   $\boldsymbol{D[a,b]}$ satisfies the vector space axioms.

3.1.2 Vector Subspaces and Spans of Finite Sets

Let H be a subset of the vector space V.

1. *Subspace:*

   H is said to be a subspace of V if the following three conditions are satisfied:

   (1) *Contains Zero Vector:* $\boldsymbol{O} \in H$.

   (2) *Closed under Addition:* If $\boldsymbol{x}, \boldsymbol{y} \in H$, then $\boldsymbol{x} + \boldsymbol{y} \in H$.

   (3) *Closed Under Scalar Multiplication:* If $\boldsymbol{x} \in H$ and $c \in \mathbf{R}$, then $c\boldsymbol{x} \in H$.

2. *Span of a Finite Set of Vectors:*

   Suppose that H is the collection of all linear combinations of a finite set of vectors in V,

   $$H = \{c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} \mid c_i \in \mathbf{R} \text{ for all } i\} \subseteq V.$$

   Then $H = \text{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ satisfies the three axioms for subspaces listed above.

Subspaces are vector spaces in their own right. Specifically,

> ***Theorem (Subspace).*** If $H \subseteq V$ satisfies the three conditions for subspaces,
>
> then H is a vector space in its own right.
>
> In particular, if $H = \text{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ is the span of a finite set of vectors, then H is a vector space in its own right.

Note: Importance of This Theorem

If H satisfies the conditions for subspaces, then H inherits all the remaining properties needed to satisfy the axioms of a full vector space. Thus, the theorem can be used to reduce the amount of work needed to determine if a collection of vectors is a vector space.

A reasonable strategy to follow to determine if H is a subspace is the following:

1. Determine if H can be written as the span of a finite set of vectors.

   If yes, then H is a subspace and a vector space in its own right.

2. If H cannot be written as the span of a finite set of vectors, then determine if H satisfies the three axioms for spaces.

   If yes, then H is a subspace and a vector space in its own right.

3. Otherwise, H is neither a subspace nor a vector space in its own right.

Further, to demonstrate that a subset is not a subspace, you just need to demonstrate that one of the three conditions fails.

Problem 1. Which, if any, of the following subsets are subspaces? Why?

(*a*) $L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, y = mx \right\} \subset \mathbf{R}^2$, where m is a fixed constant.

(*b*) $L = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, y = mx + b \right\} \subset \mathbf{R}^2$, where m and $b \neq 0$ are fixed constants.

(*c*) $D = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, x^2 + y^2 \leq 1 \right\} \subset \mathbf{R}^2$.

(d) $D = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;\middle|\; a, d \in \mathbf{R} \right\} \subset \boldsymbol{M_{2\times 2}}$.

(e) $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; d = a + b + c \right\} \subset \boldsymbol{M_{2\times 2}}$.

(f) $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; ad - bc = 0 \right\} \subset \boldsymbol{M_{2\times 2}}$.

(g) $H = \{\boldsymbol{p}(t) = a + t \mid a \in \mathbf{R}\} \subset \boldsymbol{P_3}$.

(h) $Z = \left\{\boldsymbol{p}(t) = a + bt + ct^2 + dt^3 \;\middle|\; \boldsymbol{p''}(t) = 0 \text{ for all } t\right\} \subset \boldsymbol{P_3}$.

(i) $E = \{\boldsymbol{f} : [a, b] \to \mathbf{R} \mid \boldsymbol{f} \text{ is continuous and } \boldsymbol{f}(a) = \boldsymbol{f}(b)\} \subset \boldsymbol{C[a, b]}$.

3.2 Subspaces Related to Matrices and Linear Transformations

3.2.1 Null Space and Column Space of a Matrix

Let A be an $m \times n$ matrix.

1. *Null Space of a Matrix:*

   The null space of A is the set of solutions to the matrix equation $A\boldsymbol{x} = \boldsymbol{O}$:

$$Null(A) = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{O}\} \subseteq \mathbf{R}^n.$$

2. *Column Space of a Matrix:*

   The column space of A is the set of all $\boldsymbol{b}$ for which $A\boldsymbol{x} = \boldsymbol{b}$ is consistent:

$$Col(A) = \{\boldsymbol{b} \mid A\boldsymbol{x} = \boldsymbol{b} \text{ is consistent}\} \subseteq \mathbf{R}^m.$$

For example, let $A = [\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \boldsymbol{a_3}] = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & -6 \\ 2 & 1 & 4 \end{bmatrix}$.

Since $\left[\begin{array}{c|c} A & \boldsymbol{O} \end{array}\right] = \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ -3 & 1 & -6 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$,

the null and column spaces can be written as spans as follows (please complete):

Problem 1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ -2 & -4 & 1 & 7 \end{bmatrix}$. Write $Null(A)$ and $Col(A)$ as spans of finite sets of vectors. Simplify your answers as much as possible.

3.2.2 Kernel and Range of a Linear Transformation

Let $T : V \to W$ be a function between vector spaces V and W.

1. *Linear Transformation:*

   T is said to be a linear transformation if the following two conditions are satisfied:

   (1) *T Respects Addition:* $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y})$ for every $\boldsymbol{x}, \boldsymbol{y} \in V$.

   (2) *T Respects Scalar Multiplication:* $T(c\boldsymbol{x}) = cT(\boldsymbol{x})$ for every $\boldsymbol{x} \in V$ and $c \in \mathbf{R}$.

2. *Kernel of a Linear Transformation:*

   If T is a linear transformation, then the kernel of T is the set of all vectors in the domain V that map to the zero vector in the co-domain W:

   $$Kernel(T) = \{\boldsymbol{x} \mid T(\boldsymbol{x}) = \boldsymbol{O}\} \subseteq V.$$

3. *Range of a Linear Transformation:*

   If T is a linear transformation, then the range of T is the set of all vectors in the co-domain W that can be written as the image of some vector in the domain V:

   $$Range(T) = \{\boldsymbol{b} \mid \boldsymbol{b} = T(\boldsymbol{x}) \text{ for some } \boldsymbol{x}\} \subseteq W.$$

Note 1: If $T : V \to W$ is a linear transformation, then $T(\boldsymbol{O}) = \boldsymbol{O}$

To see this (please complete),

Note 2: Linear transformations when $V = \mathbf{R}^n$ and $W = \mathbf{R}^m$

If $T : \mathbf{R}^n \to \mathbf{R}^m$ is the linear transformation with rule $T(\boldsymbol{x}) = A\boldsymbol{x}$, then

1. The kernel of T is the same as the null space of A: $Kernel(T) = Null(A)$.

2. The range of T is the same as the column space of A: $Range(T) = Col(A)$.

Problem 2. Which, if any, of the following are linear transformations? Why?

(*a*) $T : \boldsymbol{M_{2\times 2}} \to \mathbf{R}$ with rule $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

(*b*) $T : \boldsymbol{M_{2\times 2}} \to \boldsymbol{M_{2\times 2}}$ with rule $T(A) = MA$, where M is a fixed 2×2 matrix.

(c) $T : \boldsymbol{P_2} \to \boldsymbol{P_2}$, where the image of the polynomial $\boldsymbol{p}$ satisfying $\boldsymbol{p}(t) = a + bt + ct^2$ for each $t \in \mathbf{R}$, is the polynomial $T(\boldsymbol{p})$ satisfying

$$T(\boldsymbol{p})(t) = (a+1) + (b+1)t + (c+1)t^2, \text{ for each } t \in \mathbf{R}.$$

(d) $T : \boldsymbol{M_{2\times 2}} \to \boldsymbol{M_{2\times 2}}$ with rule $T(A) = A + A^T$.

Problem 3. The following three functions are linear transformations. In each case, give explicit descriptions of the kernel and range of the transformation.

(*a*) $T : \boldsymbol{P_3} \to \boldsymbol{P_3}$, where the image of $\boldsymbol{p}$ is its derivative, $T(\boldsymbol{p}) = \boldsymbol{p'}$.

(*b*) $T : \mathbf{R}^2 \to \boldsymbol{M_{2\times 2}}$ is the function with rule $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a+b & 2b \\ -b & a \end{bmatrix}$.

(c) $T : \boldsymbol{P_3} \to \mathbf{R}^2$ is the function with rule $T(\boldsymbol{p}) = \begin{bmatrix} \boldsymbol{p}'(0) \\ \boldsymbol{p}''(0) \end{bmatrix}$.

3.3 Linear Independence and Bases

3.3.1 Linearly Independence and Linear Dependence

Let V be a vector space. The subset $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\} \subseteq V$ is said to be *linearly independent* when the linear combination $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k}$ equals the zero vector only when all scalars c_i are zero. That is, when

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} = \boldsymbol{O} \;\Rightarrow\; c_1 = c_2 = \cdots = c_k = 0.$$

Otherwise, the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ is said to be *linearly dependent.*

Note that:

1. If $\boldsymbol{v_i} = \boldsymbol{O}$ for some i, then the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ is linearly dependent.

2. If each $\boldsymbol{v_i} \neq \boldsymbol{O}$, then the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ is linearly dependent if and only if one vector can be written as a linear combination of the others.

Problem 1. In each case, determine if the set is linearly independent or linearly dependent. If linearly dependent, find a dependence relationship among the $\boldsymbol{v_i}$'s.

$$(a)\ \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} = \left\{ \begin{bmatrix} 1 \\ 4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ -1 \\ 9 \end{bmatrix} \right\} \subseteq \mathbf{R}^4.$$

(*b*) $\{\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}\} \subseteq \boldsymbol{P_2}$, where the functions satisfy

$$\boldsymbol{p_1}(t) = 1 - t,\ \boldsymbol{p_2}(t) = 1 - t^2, \text{ and } \boldsymbol{p_3}(t) = 1 + 2t - 3t^2 \text{ for all } t \in \mathbf{R}.$$

(c) $\{M_1, M_2, M_3\} = \left\{ \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \right\} \subseteq \boldsymbol{M_{2\times 2}}$.

3.3.2 Basis of a Vector Space

A *basis* for the vector space V is a linearly independent set that spans V. Specifically, the set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ is said to be a basis for V if

1. $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$ is linearly independent and
2. $V = \text{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}\}$.

Problem 1, continued. Using the work above,

(a) $V = \text{Span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} \subseteq \mathbf{R}^4$ has basis ________________________________.

(b) $V = \text{Span}\{\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}\} \subseteq \boldsymbol{P_2}$ has basis ________________________________.

(c) $V = \text{Span}\{M_1, M_2, M_3\} \subseteq \boldsymbol{M_{2\times 2}}$ has basis ____________________________.

Problem 2. In each case, find a basis for V.

(a) $V = \text{Span}\left\{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}\right\} \subseteq \boldsymbol{M_{2\times 2}}$.

(b) $V = \{A \ : \ \text{Each column of } A \text{ has sum } 0\} \subseteq \boldsymbol{M_{2\times 3}}$.

3.3.3 Bases for Null and Column Spaces

Let A be an $m \times n$ matrix. Then

1. The method we use to write $Null(A)$ as the span of a finite set of vectors automatically produces a linearly independent set, hence a basis for $Null(A)$.
2. The pivot columns of A form a basis for $Col(A)$.

Problem 3. Let $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \end{bmatrix}$. Use the fact that

$$[\ A \mid \boldsymbol{O}\ ] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 0 \\ 2 & 4 & -1 & 3 & 0 \\ 3 & 6 & 2 & 22 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 2 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

to find bases for $Null(A)$ and $Col(A)$.

3.4 Unique Representations and Coordinate Systems

3.4.1 Unique Representation Theorem

Bases can be used to establish coordinate systems. To begin, we consider the following theorem:

> ***Unique Representation Theorem.*** Let $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_k}\}$ be a basis for the vector space V. Then for each $\boldsymbol{v} \in V$,
>
> $$\boldsymbol{v} = c_1\boldsymbol{b_1} + c_2\boldsymbol{b_2} + \cdots + c_k\boldsymbol{b_k} \text{ for unique constants } c_1, c_2, \ldots, c_k.$$
>
> The c_i's are called the *coordinates* of $\boldsymbol{v}$ in basis $\mathcal{B}$.

<u>*Proof:*</u>

The proof of the unique representation theorem is straightforward. Suppose that

$$\boldsymbol{v} = c_1\boldsymbol{b_1} + c_2\boldsymbol{b_2} + \cdots + c_k\boldsymbol{b_k} = d_1\boldsymbol{b_1} + d_2\boldsymbol{b_2} + \cdots + d_k\boldsymbol{b_k}$$

for scalars $c_i, d_i \in \mathbf{R}$. Then, by subtracting the second representation from the first, we get

$$\boldsymbol{O} = (c_1 - d_1)\boldsymbol{b_1} + (c_2 - d_2)\boldsymbol{b_2} + \cdots + (c_k - d_k)\boldsymbol{b_k}.$$

Now, since $\mathcal{B}$ is a basis, we know that

$$c_i - d_i = 0 \text{ (equivalently, } c_i = d_i) \text{ for each } i.$$

Thus, the representation is unique.

Coordinate Vectors. Let $\boldsymbol{v} = c_1\boldsymbol{b_1} + c_2\boldsymbol{b_2} + \cdots + c_k\boldsymbol{b_k}$ be the unique representation of $\boldsymbol{v}$ in the basis $\mathcal{B}$. Then

$$[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbf{R}^k \quad \text{is the } \textit{coordinate vector} \text{ of } \boldsymbol{v} \text{ in the basis } \mathcal{B}.$$

For example, let $\mathcal{E} = \{\boldsymbol{e_1}, \boldsymbol{e_2}\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ be the standard basis of $\mathbf{R}^2$.

Further, let $\boldsymbol{v} = \begin{bmatrix}x\\y\end{bmatrix}$ be an arbitrary vector in $\mathbf{R}^2$. Since $\boldsymbol{v} = x\boldsymbol{e_1} + y\boldsymbol{e_2}$, the coordinate vector of $\boldsymbol{v}$ in the standard basis is the vector itself: $[\boldsymbol{v}]_{\mathcal{E}} = \boldsymbol{v}$.

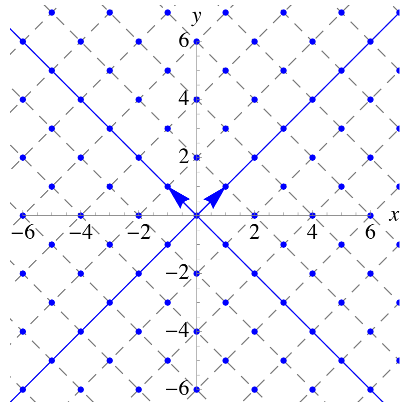
Problem 1. Consider the basis

$$\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

of $\mathbf{R}^2$, and let $\boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ be a vector.

Find the coordinate vector of $\boldsymbol{v}$ in basis $\mathcal{B}$, $[\boldsymbol{v}]_{\mathcal{B}}$.

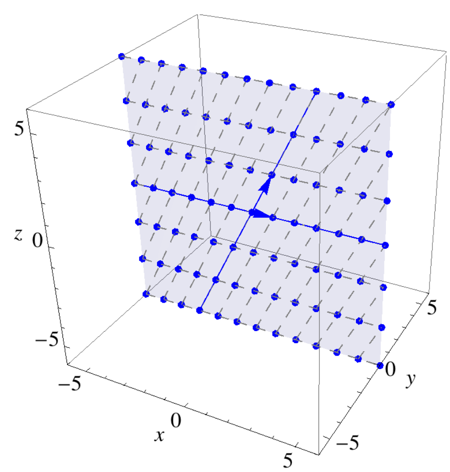
[The plot shows a grid of linear combinations of the form $c\boldsymbol{b_1} + d\boldsymbol{b_2}$, where either c or d is an integer.]



Problem 2. $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ is a basis for a subspace V of $\mathbf{R}^3$.

Let $\boldsymbol{v} = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} \in V$. Find $[\boldsymbol{v}]_{\mathcal{B}}$.

[V corresponds to the xz-plane in 3-space. The plot shows a grid of linear combinations of the form $c\boldsymbol{b_1} + d\boldsymbol{b_2}$, where either c or d is an integer.]



Problem 3. $\mathcal{B} = \{1 - t^2, 1 + t^2\}$ is a basis for a subspace V of $\boldsymbol{P_2} = \{a + bt + ct^2 \ : \ a, b, c \in \mathbf{R}\}$. Let $\boldsymbol{p}(t) = 2 + 7t^2$. Find $[\boldsymbol{p}]_{\mathcal{B}}$.

3.4.2 Change of Coordinates in k-Space

Let $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_k}\}$ be a basis for $\mathbf{R}^k$ and $P_{\mathcal{B}}$ be the matrix whose columns are the $\boldsymbol{b_i}$'s:

$$P_{\mathcal{B}} = \begin{bmatrix} \boldsymbol{b_1} & \boldsymbol{b_2} & \cdots & \boldsymbol{b_k} \end{bmatrix}.$$

Since the columns of $P_{\mathcal{B}}$ are linearly independent, we know that $P_{\mathcal{B}}$ is invertible and

$$P_{\mathcal{B}}\, \boldsymbol{c} = \boldsymbol{v} \implies \boldsymbol{c} = P_{\mathcal{B}}^{-1}\boldsymbol{v} \text{ or } [\boldsymbol{v}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\boldsymbol{v}.$$

For this reason, $P_{\mathcal{B}}$ is called the *change-of-coordinates matrix.*

Problem 1, continued. As above, let $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ and $\boldsymbol{v} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Find $P_{\mathcal{B}}$, and verify that $[\boldsymbol{v}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\boldsymbol{v}$ is the same as the vector obtained earlier.

3.4.3 Coordinate Mapping Theorem

The process of finding coordinate vectors can be viewed as a linear transformation. Specifically,

> ***Coordinate Mapping Theorem.*** Let $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_k}\}$ be a basis for the vector space V. The function
>
> $$T : V \to \mathbf{R}^k \text{ with rule } T(\boldsymbol{v}) = [\boldsymbol{v}]_{\mathcal{B}}$$
>
> is a linear transformation that is both one-to-one and onto. The linear transformation T is known as the *coordinate mapping function.*

For example, $\mathcal{B} = \{1, t, t^2\}$ is the standard basis for $\boldsymbol{P_2} = \{a + bt + ct^2 \ : \ a, b, c \in \mathbf{R}\}$. With respect to this basis, the coordinate mapping function is the transformation

$$T : \boldsymbol{P_2} \to \mathbf{R}^3 \text{ which maps } \boldsymbol{p}(t) = a + bt + ct^2 \text{ to } T(\boldsymbol{p}) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Isomorphisms; Isomorphic vector spaces. An *isomorphism* is an invertible linear transformation $T : V \to W$ between vector spaces V and W. If an isomorphism exists, then we say that the vector spaces V and W are *isomorphic* (have the same structure).

The coordinate mapping theorem tells us that V and $\mathbf{R}^k$ are isomorphic vector spaces. The practical implication of this theorem is that a problem involving vectors in V can be translated to $\mathbf{R}^k$ (via the coordinate mapping function), solved in $\mathbf{R}^k$ and translated back.

Continuing with the example above, suppose that we are interested in determining if the set

$$\{\boldsymbol{p_1}(t), \boldsymbol{p_2}(t), \boldsymbol{p_3}(t)\} = \{1 + 2t^2, 4 + t + 5t^2, 3 + 2t\} \subset \boldsymbol{P_2}$$

is linearly independent. We could equivalently determine if

$$\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\} \text{ is a linearly independent set in } \mathbf{R}^3.$$

Since $\left[\ \boldsymbol{v_1} \quad \boldsymbol{v_2} \quad \boldsymbol{v_3} \mid \boldsymbol{O}\ \right] =$

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

we know that the set of vectors in $\mathbf{R}^3$ and the set of pre-images in $\boldsymbol{P_2}$ are linearly dependent.

Further, we can write $\boldsymbol{p_3}(t)$ as a linear combination of the first two polynomials as follows:

$\boldsymbol{p_3}(t) =$ ____ for all $t \in \mathbf{R}$.

Problem 4. $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is the standard basis for $\boldsymbol{M_{2\times 2}}$. With respect to this basis, the coordinate mapping function is the transformation

$$T : \boldsymbol{M_{2\times 2}} \to \mathbf{R}^4 \text{ with rule } T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Using the standard basis and coordinate mapping function,

(*a*) Determine if

$$\{M_1, M_2, M_3\} = \left\{ \begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix} \right\}$$

is a linearly independent set in $\boldsymbol{M_{2\times 2}}$. If the set is linearly dependent, express one of the matrices as a linear combination of the others.

(b) Demonstrate that $\{A_1, A_2, A_3, A_4\} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ is a linearly independent set in $\boldsymbol{M_{2\times 2}}$.

(c) Express $\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ as a linear combination of the 4 matrices given in part (b).

3.5 Dimension and Rank

3.5.1 Dimension of a Vector Space

The following theorem tells us that if a vector space V has a basis with k elements, then all bases of V must have exactly k elements.

> ***Theorem (Finite Bases).*** If $\mathcal{B} = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \ldots, \boldsymbol{b_k}\}$ is a basis for V, then
>
> 1. Any subset of V with more than k vectors must be linearly dependent.
> 2. Every basis of V must contain exactly k vectors.
> 3. Every linearly independent set with exactly k vectors is a basis of V.

Note that each part of this theorem can be proven using the coordinate mapping theorem.

k-dimensional vector spaces: The vector space V is said to be *k-dimensional* if V has a basis with k elements. In this case, we write $dim(V) = k$ ("the dimension of V equals k").

For example, given positive integers m and n,

$$dim(\mathbf{R}^n) = \_\_\_\_\_\_\_\_,\ dim(\boldsymbol{P_n}) = \_\_\_\_\_\_\_\_ \text{ and } dim(\boldsymbol{M_{m\times n}}) = \_\_\_\_\_\_\_\_.$$

Trivial vector space: By convention, the dimension of the trivial vector space is zero,

$$dim(\{\boldsymbol{O}\}) = 0.$$

Note that the trivial vector space does not have a basis since $\{\boldsymbol{O}\}$ is linearly dependent.

Infinite-dimensional vector spaces: If V is not trivial and is not spanned by a finite set of vectors, then V is said to be *infinite-dimensional.*

Examples of infinite-dimensional vector spaces include

1. The vector space of polynomials of all orders,

   $$\boldsymbol{P_\infty} = \{\boldsymbol{p}(t) \mid \boldsymbol{p}(t) \text{ is a polynomial function of any degree}\},$$

   with the usual definitions of addition and scalar multiplication.
2. The vector space of continuous real-valued functions on the interval $[a, b]$,

   $$\boldsymbol{C[a,b]} = \{\boldsymbol{f} : [a,b] \to \mathbf{R} \mid \boldsymbol{f} \text{ is continuous}\},$$

   with the usual definitions of addition and scalar multiplication.

3.5.2 Dimension of a Vector Subspace

The dimension of a subspace of V can never be more than the dimension of V. Specifically,

> ***Theorem (Subspaces).*** If $H \subseteq V$ is a subspace of a vector space with finite dimension k, then H has finite dimension and $dim(H) \leq dim(V) = k$.

Problem 1. In each case, find the dimension of the subspace H.

$$(a)\ \ H = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} \ :\ a,b,c,d \in \mathbf{R} \right\} \subseteq \mathbf{R}^4.$$

(*b*) $H = \text{Span}\{\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3}\} \subseteq \boldsymbol{P_2}$, where

$$\boldsymbol{p_1}(t) = t + t^2,\ \boldsymbol{p_2}(t) = -1 + t,\ \boldsymbol{p_3}(t) = -1 + 4t + 3t^2,\ \text{for each } t \in \mathbf{R}.$$

3.5.3 Dimensions of Null and Column Spaces of a Matrix

Let A be an $m \times n$ matrix. The dimension of the null space corresponds to the number of free variables when solving $A\boldsymbol{x} = \boldsymbol{O}$. The dimension of the column space corresponds to the number of pivot columns of A.

Problem 2. In each case, find the dimensions of the null and column spaces of A.

$$(a)\ A = \begin{bmatrix} 1 & -6 & 0 & 0 & -20 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $A = \begin{bmatrix} -1 & -3 & 2 & 6 \\ 2 & 6 & -5 & -12 \\ 1 & 3 & -3 & -6 \end{bmatrix}$.

3.5.4 Rank and Nullity of a Matrix and its Transpose

Let A be an $m \times n$ matrix. Then

1. *Rank of a Matrix:*
   The rank of A is the dimension of its column space: $rank(A) = dim(Col(A))$.
2. *Nullity of a Matrix:*
   The nullity of A is the dimension of its null space: $nullity(A) = dim(Null(A))$.

Similarly, if A^T is the $n \times m$ transpose of the matrix A, then we let

$$rank(A^T) = dim(Col(A^T)) \text{ and } nullity(A^T) = dim(Null(A^T)).$$

The following theorem gives interesting relationships among these numbers:

Rank Theorem. Let A be an $m \times n$ matrix, and let A^T be $n \times m$ the transpose of the matrix A. Then

1. $rank(A) + nullity(A) = n$.
2. $rank(A^T) + nullity(A^T) = m$.
3. $rank(A^T) = rank(A)$.

Problem 3. Explain why the first part of the theorem is true.

Problem 4. Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

Find $rank(A)$, $rank(A^T)$, $nullity(A)$, $nullity(A^T)$.

Problem 5. Let I_n be the $n \times n$ identity matrix and $O_{m\times n}$ be the $m \times n$ zero matrix.

Fill-in the following table:

	Size of A: $m \times n$	$rank(A) = rank(A^T)$	$nullity(A)$	$nullity(A^T)$
$A = \begin{bmatrix} I_3 & O_{3\times 2} \end{bmatrix}$				
$A = \begin{bmatrix} I_3 & I_3 \\ O_{2\times 3} & O_{2\times 3} \end{bmatrix}$				
$A = O_{3\times 2}$				

Problem 6. Let A be an $m \times n$ matrix. In each case, determine if the statment is TRUE or FALSE, and give a reason for each choice.

(a) $Col(A) = Col\left(\begin{bmatrix} A \\ A \end{bmatrix}\right)$.

(b) $Null(A) = Null\left(\begin{bmatrix} A \\ A \end{bmatrix}\right)$.

(c) $Col\left(A^T\right) = Col\left(\begin{bmatrix} A \\ A \end{bmatrix}^T\right)$.

(d) $Null\left(A^T\right) = Null\left(\begin{bmatrix} A \\ A \end{bmatrix}^T\right)$.